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The Flow of Cohesionless Grains in Fluids

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Phil. Trans. R. Soc. Lond. A 1956 **249**, 235-297
doi: 10.1098/rsta.1956.0020

THE FLOW OF COHESIONLESS GRAINS IN FLUIDS

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(*Received* 31 *January* 1955—*Revised* 14 *April* 1956)

[Plate 2]

CONTENTS

Part I. The results of previous experiments (Bagnold 1954) on the stresses set up in a uniform gravity-free dispersion of solid grains when uniformly sheared in a fluid are applied to the non-uniform case of grain flow over a gravity bed, assuming the results are quantitatively applicable to any sufficiently thin shear layer.

It is found that if the bed is composed entirely of potentially mobile grains a stress-equilibrium relation at the bed surface can be defined whereby the magnitude of a certain 'bed load' of grains in transit over unit bed area is given in terms of the applied tangential stress. The bed load is independent both of the existence of any additional suspended load and of the degree of dispersion of the grains. The state of internal fluid motion enters as a single experimental constant.

From a consideration of the stability of this equilibrium relation it is possible to predict the conditions under which an initially plane bed surface should become rippled; and general quantitative agreement is found with experimental data both for wind-blown and water-driven grains. Primary and secondary bed rippling are distinguished. The magnitude of the 'form-drag' due to primary bed ripples can be calculated. That due to secondary ripples is definable as an experimental constant.

The gravity-free experiments disclosed that the shear resistance of a grain dispersion may vary as the square or the first power of the rate of shear, analogously to that of a true fluid, according to the value of a number G analogous to a Reynolds number. The square law is followed when the effects of grain inertia dominate over those of fluid viscosity. Assuming that the phenomenon of 'saltation' as observed over a gravity bed is an inertia effect, the conditions for saltation are predictable. The results again agree quantitatively with observation.

Part II. The resistance offered by the grains to their displacement along the flow is shown to be proportional to their normal immersed weight component. And their measurable mass transport rate is hereby proportional to the rate of useful work done in transporting them. On this basis separate expressions are found for the transport rates of the bed load and suspended load, in terms of the applied tangential stress and of a tangential and a normal relative velocity respectively.

When conditions are restricted to those of the 'stream case' these velocities become constant for any given system, being functions of an appropriate constant mean drag coefficient.

The bed-load transport-rate expression found gives magnitudes, and variations of magnitude with grain size, in agreement with the experimental data for wind-blown sand. Agreement is also found for water-driven grains in open channels from the threshold of movement up to a certain value of the applied stress. The experimental rates are then found to increase suddenly. This increase is attributed to the development of an additional suspended load. The abrupt development of a suspended load may be explained as due to a change in the nature of the fluid turbulence when the stationary boundary becomes occluded from the fluid flow by a concentrated layer of moving bed-load grains.

The assumption that under these new moving-boundary conditions the available fluid energy derived from shearing over the bed is equally apportioned between bed-load and suspended-load transport work leads to values for the suspended-load transport rate which agree closely with the experimental data.

A critical relation emerges between the gravity slope of the bed, the fall velocity and the mean transport velocity of the suspended grains at which their transport may become very large.

Conditions are examined under which the steady transport may be possible of grains of heterogeneous size or density.

Part III. When the fluid flow is non-inertial (laminar) and the grain flow is also non-inertial the semi-empirical relations found previously for the internal stresses are such that both viscosity and shear rate can be eliminated, and a differential equation obtained whose solution gives the grain concentration in terms only of distance from the bed and of the applied tangential stress. It appears that with constant applied stress (unlimited flow depth) the degree of grain dispersion greatly exceeds that to be expected in turbulent fluid flow. But when the applied stress diminishes linearly with distance from the bed boundary a possible solution gives constant grain concentration throughout the flow. This appears to explain certain experimental results, including the behaviour of 'slurries'.

The effect is examined of a fixed or partially fixed bed on the grain flow in a turbulent fluid. The effect may be pronounced in the case of suspended grains.

Under certain clearly definable conditions a loose grain bed must cease to remain stationary. And if the fluid flow above is turbulent the whole grain bed should flow at constant maximum concentration, underneath the flow proper and separated from it by a moving-bed surface interface at which the concentration is discontinuous. This explains phenomena sometimes found under river torrents.

The factors giving rise to and limiting the development of bed features (dunes) on a bigger scale than ripples are examined. Dune formation appears as an inherent tendency of the grain flow alone, which may or may not be inhibited by the conditions of the fluid flow.

List of principal symbols

In the third column '0' signifies dimensionless quantity.

symbol	introduced in section		meaning
A	7*a*	0	numerical transport-rate coefficient; found to be a general constant for the 'stream case'
A'	8*a*	0	ditto, experimentally variable; $A' = AB$
B	7*a*	0	the group of dimensionless parameters in Φ
C	1*b*	0	grain concentration by volume $= \dfrac{\text{grain-occupied space}}{\text{whole space}}$
D	1*b*		grain diameter
G	1*b*	0	grain-flow number involving grain shear stress T
h	3*a* (iii)	0	$= y/D$
N	1*b*	0	grain-flow number involving rate of grain shear $\mathrm{d}U/\mathrm{d}y$

P	1 *b*		dispersive stress between sheared grains, acting normally to shear planes
p	3 *a* (iii)		static fluid pressure
Q	6 *a*		grain transport rate = mass passing per unit flow width in unit time
$\mathscr{T}$	1 *b*		overall (measurable) tangential stress $T+\tau'$
$\mathscr{T}_F$	3 *a* (iii)		apparent applied tangential stress due to action of external forces on fluid only
T	1 *b*		component of $\mathscr{T}$ due to influence of grain on grain
U	1 *b*		absolute grain velocity in direction of flow
$\overline{U}$	7 *c*		mean travel velocity of grains
$\mathscr{U}$	6 *c*		grain velocity relative to fluid in flow direction
$\overline{\mathscr{U}}$	6 *d*		mean value of $\mathscr{U}$ relevant to mean drag coefficient $\overline{\psi'}$
u	7 *b*		fluid velocity in direction of flow
$\mathscr{V}$	6 *c*		grain velocity relative to fluid, normal to bed
W	7 *a*		rate of useful work done in transporting grains
W_F	7 *a*		ditto by fluid only, excluding tangential gravity component on grains
x	3 *a* (i)		direction of flow
y	1 *b*, 3 *a* (i)		normal distance above flow boundary
α	1 *b*	0	'friction angle'; $\tan\alpha$ = grain-stress ratio T/P
β	3 *a* (i)	0	declination of bed surface below horizontal
γ	3 *a* (ii)		unit stress $(\sigma-\rho)\, gD\cos\beta$
η	1 *b*		fluid viscosity
η'	1 *b*		apparent viscosity of intergranular fluid
Θ	3 *a* (iii)	0	overall tangential stress in terms of $\gamma = \mathscr{T}/\gamma$
Θ_F	3 *b*	0	$=\mathscr{T}_F/\gamma$
$\Theta_{(t)}$	3 *c*	0	threshold value of Θ_0 at which grains begin to move over a plane grain bed
Θ_t	4 *c*	0	general threshold value; covers a new extrapolated threshold in case of secondary ripples
λ	1 *b*	0	linear grain concentration; a direct function of C
ν	2 *a*		kinematic viscosity
ρ	3 *a* (ii)		fluid density
σ	1 *b*		grain density
τ	1 *b*		tangential stress in a grainless fluid
τ'	1 *b*		ditto, modified by presence of dispersed grains; component of $\mathscr{T}$ due only to distortion of the intergranular fluid
τ''	4 *a*		fluid stress due to form-drag of primary ripples
Φ	7 *a*	0	transport-rate function $=\Phi'/B$
Φ'	7 *a*	0	the group $Q/\sigma D\surd(\gamma/\rho)$ of dimensional quantities in Φ
χ	3 *a* (iv)	0	normal immersed weight component of grains, in terms of γ
ψ	6 *b*	0	drag coefficient for a single isolated grain
ψ'	6 *b*	0	ditto, increased value due to proximity of other grains

General suffixes

—	constant static values at negative distances below bed surface
o	values above but immediately adjacent to bed surface
b	refers to bed load
s	refers to suspended load
a	refers to an upper plane of zero shear
f	refers to an underlying plane of a fixed floor beneath a grain bed

PART I. GRAIN FLOW OVER A GRAVITY GRAIN BED; GENERAL IDEAS

1. Introduction

(a) *Scope and approach*

The flow of solid grains consists essentially of their general shearing over one another and over a stationary boundary or bed, in the presence of a fluid which partakes of the shearing. A body-force component pulls them towards the bed; and a tangential force maintains their shearing motion.

The tangential force on the grains may consist largely of a body-force component, as in the avalanching of granular matter down a slope. Or it may be almost exclusively a fluid force due to the flow of the fluid, as in the driving of grains over the bed by a stream of wind or of water. Here the grain concentration is found to attenuate progressively to zero with increasing height above the bed. The special case of this when the fluid is turbulent and when the bed consists of loose grains is of particular practical importance, and will for convenience be called 'the stream case'.

A full range of intermediate combinations is possible, with or without fluid turbulence and with or without a grain bed. The case of grain transport through pipes can be fitted into this broad scheme by regarding the pressure gradient along the pipe as an extra body-force in the direction of flow acting on the volumes of grains and fluid alike irrespective of density.

Empirical knowledge exists. But it is limited to rather narrow and isolated ranges of practical conditions covered mainly by the 'stream case' only. No general underlying principles having emerged, it has not been possible to predict anything useful about the bulk movement of grains under unexperienced conditions, or to apply experimental knowledge obtained from one set of conditions to another different set.

No purely theoretical approach to the discovery of such general principles seems possible at present because insufficient is known about the fluid to define how it is affected by the presence within it of solid grains in appreciable concentration. And further complications are introduced by a deformable grain boundary to the flow, the shape of the boundary becoming moulded by the flow into ripples and dunes which in turn modify the flow.

The present approach is therefore necessarily empirical. Certain general ideas and relations will be considered in part I. These will be applied in part II to the special conditions of the 'stream case' to which most of the available quantitative data are confined. Conditions bordering the stream case and therefore limiting the applicability of the results obtained in part II will be examined in part III.

Clues to underlying principles are often forthcoming from a recognition of certain simple things found common to all aspects of a phenomenon. In the case of grain flow one such clue is obvious. A static grain mass cannot be sheared without some degree of dilation or dispersion. And since in this case the dispersion must be upwards, against the bed-ward body force, all grain flow requires the exertion of some kind of dispersive stress normally between the sheared grain layers and between them and the bed.

A second clue comes from Einstein's (1906) work on the apparent viscosity of a fluid containing a dispersion of solid spheres. In this and subsequent work two influences due to the presence of the grains were recognized: that of the grains on the fluid flow, and that of the grains upon one another. The former, being capable of mathematical definition, was dealt with alone by confining the conditions to very low grain concentrations at which the grains were supposed not to influence one another.

But when the grains are but a diameter apart or less (volume concentration $>9\%$ for spheres) the probability of mutual encounter, always finite for a finite mean concentration of a random grain array undergoing shear, approaches a certainty. The grains must knock or push each other out of the way according to whether or not the effects of their inertia outweigh those of the fluid's viscosity; and both kinds of encounter must involve displacements of the grains normally to the planes of shear. So it seemed probable that the required normal dispersive stress between the sheared grains might arise from the neglected influence of grain on grain. Further, the same encounters should give rise to an associated shear resistance additional to that offered by the intergranular fluid.

The existence of these two novel kinds of stress has been confirmed by an experimental study (Bagnold 1954) of the dynamics of grain dispersions sheared in the absence of any applied body force. Since the results form the basis of the present theory, a summary follows immediately.

(*b*) *Dynamics of a sheared grain dispersion* (*summary*)

From the still imperfect theory it will be assumed that:

(i) The overall (measurable) shear resistance $\mathscr{T}$ can be written

$$\mathscr{T} = T + \tau', \tag{1}$$

where T and τ' are the resisting shear-stress elements due respectively to the encounters between grains, and to the distortion of the intergranular fluid alone, as modified by the presence of the grains. T and τ' can be regarded as discrete quantities, T pertaining only to the disperse phase and τ' only to the continuous phase.

(ii) The normal dispersive stress P between the grains, which is due to the same encounters, is related to T by

$$T/P = \tan\alpha, \tag{2}$$

where the angle α is associated with the mean encounter conditions, and $\tan\alpha$ is therefore the dynamic analogue of the static friction coefficient.

(iii) Irrespectively of whether the shearing of the intergranular fluid is inertial (turbulent) or otherwise, the shearing of the grain dispersion within it, regarded as a separate fluid, may also be inertial or otherwise, the inertia being that of the grains. So, depending on the fluid viscosity, the grains' size and density, and the rate of shear, the stresses T and P pertaining to the dispersion only may vary as the square or the first power of the rate of shear just as in the case of the real fluid.

(iv) When the grain shearing is inertial and the encounters become collisions, the grain stresses T and P arise from the diffusion of tangential and normal components of grain momenta created at the collisions. Hence the normal stress P may increase by a factor of 2 as between inelastic and elastic collisions, with a consequent reduction in the value of the stress ratio $\tan\alpha$ for elastic grains. Most natural fluid-driven grains can, however, probably be taken as elastic, for otherwise they would not have endured as grains.

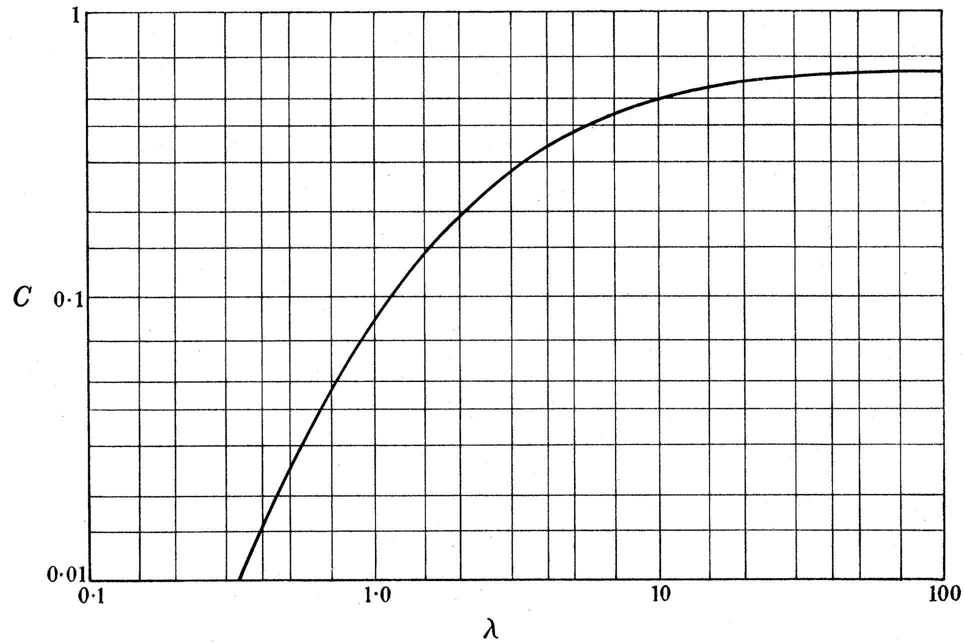

Figure 1. Relation between linear concentration, λ, and volume concentration, C, for $C_* = 0{\cdot}65$, where C_* = maximum concentration of natural grains at rest and $\lambda = \dfrac{1}{(C_*/C)^{\frac{1}{3}}-1}$.

(v) The concentration enters in a linear form

$$\lambda = \frac{\text{grain diameter}}{\text{mean radial separation distance}}.$$

This is related geometrically to the volume concentration C by

$$\lambda = \frac{1}{(C_*/C)^{\frac{1}{3}}-1} \quad \text{or} \quad C = \frac{C_*}{(1/\lambda+1)^3}, \tag{3}$$

where C_* is the maximum possible value of C when all the grains are in static contact ($= 0{\cdot}74$ for uniform spheres perfectly piled). A lower value 0·65 will be assumed for natural reasonably rounded and uniform grains. For this value of C_* relation (3) is plotted in figure 1. General shearing becomes theoretically possible when λ is reduced to 22 from its infinite static value.

The experimental system consisted of uniform spheres, assumed elastic, made free of all effective body forces by immersion in a liquid of exactly the same density. The mixture was

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sheared in the annular space between two coaxial drums, the outer of which could be rotated at a range of known speeds. The concentration was varied from $C = 0{\cdot}21$ upwards ($\lambda > 1{\cdot}3$), and the fluid viscosity η was varied between 0·01 (water) and 0·07, with check tests at the far higher viscosity of pure glycerine. The arrangement was such that the normal stress P could be measured as the inward radial pressure on the wall of the inner drum, and the overall shear stress $\mathscr{T}$ as the torque on it. Both measurements could be compared at the same rate of shear with the values obtaining at zero concentration (fluid only).

At concentrations just below 22 the substance behaved as a granular paste. But as λ was decreased the residual shear resistance at zero rate of shear diminished rapidly, till at about $\lambda = 14$ it was undetectable. At all lower concentrations the grain dispersion itself behaved as a Newtonian fluid.

At zero concentration $\mathscr{T}$ becomes the conventional fluid shear resistance τ. When in the absence of any grains τ was due to fluid turbulence, at high rates of shear the rise in $\mathscr{T}$ with increase of concentration at constant shear rate was such that at the 'fluid' limit $\lambda = 14$ the overall shear resistance $\mathscr{T}$ exceeded 100τ. Hence since any increase in the resistance of the substance to internal shear must diminish the intensity of the turbulence, $\mathrm{d}\tau'/\mathrm{d}\lambda$ should be negative. And it seems clear that at high concentrations the turbulence may disappear altogether, so that $\tau' \ll \tau \ll \mathscr{T}$.

Unfortunately, it was not found possible to measure the separate value of τ'. So the quantitative effect on the intensity of turbulence of the presence of a grain dispersion within the fluid is still unknown.

When, on the other hand, at low rates of shear τ for the grainless fluid was due only to viscosity, any reasonable extrapolation of Einstein's relation $\eta'/\eta = 1 + \frac{5}{2}C$ would at constant rate of shear make $\mathrm{d}\tau'/\mathrm{d}\lambda$ positive. Since the influence of grain on grain has now already been excluded as T, a direct extrapolation of this relation to high concentrations may not be very much in error.† At the limiting concentration the fluid resistance τ' is thereby increased by a factor of 2·5. But the measured value of $\mathscr{T} = T + \tau'$ was in this case found to exceed 120τ.

In both cases, therefore, the residual fluid shear stress element τ' becomes insignificant at high concentrations compared to the grain element T, and nearly the whole of the overall stress $\mathscr{T}$ is due to the grain encounters.

In view of this it was possible by making a small arbitrary adjustment for τ' at the lower end of the experimental range of $\mathscr{T}$ to obtain values of T to a reasonable degree of accuracy.

All the experimental values of the grain stresses T and P for $\lambda < 14$ obtained by varying λ, η, and the rate of grain shear $\mathrm{d}U/\mathrm{d}y$ were found to conform to a pair of single-valued relationships between two dimensionless numbers

$$N = \frac{\lambda^{\frac{1}{2}}\sigma D^2\,\mathrm{d}U/\mathrm{d}y}{\eta} \quad \text{and} \quad G = \frac{D}{\eta}\sqrt{\left(\frac{\sigma}{\lambda} \times \text{grain stress}\right)}, \tag{4}$$

† As a crude alternative estimate of the increase in fluid shear resistance with concentration at constant shear rate in the viscous case, the grain content C may be transformed into parallel laminae of equivalent volume. $\eta'/\eta = \tau'/\tau$ is then given at once as the ratio $1/(1-C)$ by which the shear rate of the interlaminar fluid is increased. The numerical discrepancy between the two expressions never exceeds a factor of 1·225 attained at $C = 0{\cdot}3$; and both give equal values of 2·5 at $C = 0{\cdot}6$, which for spheres corresponds to the limiting 'fluid' concentration $\lambda = 14$.

where σ and D are the grains' density and diameter. N and G have forms analogous to velocity and stress Reynolds numbers. The stress relationships in terms of N and G are shown together in figure 2. This figure epitomizes the experimental results and will be referred to repeatedly.

It will be seen from the figure that at high rates of shear in terms of N the effects of grain inertia at the encounters make the stresses follow the square law; whereas at low rates of shear they follow the linear law. Within the experimental range of λ the empirical expressions for T in the two extreme regions are

$$T_{\text{inertial}} = 0{\cdot}013\sigma(\lambda D)^2\,(\mathrm{d}U/\mathrm{d}y)^2, \quad T_{\text{viscous}} = 2{\cdot}2\lambda^{\frac{3}{2}}\eta\,\mathrm{d}U/\mathrm{d}y. \tag{5}$$

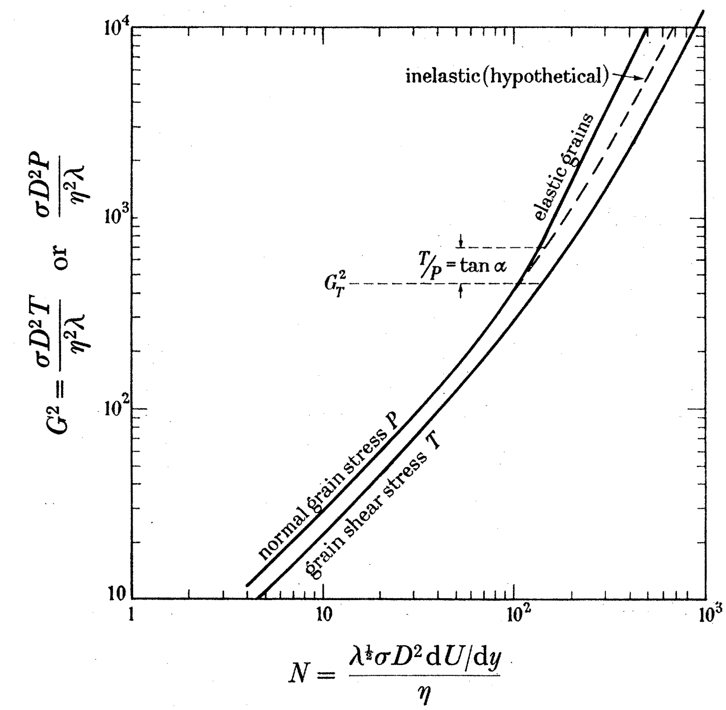


FIGURE 2. Experimental relation between shear stress T carried by the grains of a sheared dispersion, and the normal dispersive stress P between the grains, in terms of the dispersion numbers G and N. σ = grain density; D = grain diameter; λ = linear concentration; η = fluid viscosity.

The transition from the inertial to the non-inertial or viscous state of grain shearing appears to be uniquely definable for either stress by the values of N or of G according to figure 2. The lower limit of the transition in terms of G_T^2 is seen to be about 100 for both stresses. But probably owing to the elastic effect the upper limits differ, being about 3000 for T and 1000 for P. Since both stresses are equally relevent in what follows the mean upper limit will be taken as 1500. These critical values of G_T^2 will be referred to in connexion with 'saltation'.

The value of the stress ratio $T/P = \tan\alpha$ can be read off the figure for any given value of G_T^2. It approaches a small constant figure of about 0·32 for fully inertial shearing, and reaches another larger constant figure of 0·75 when viscous effects dominate. The static value $\sim$0·63 of $\tan\alpha$, as given approximately by the limiting angle $\sim$32° of repose for most reasonably sounded natural sands, and as likely to obtain within a gravity grain bed, is intermediate between the above two limiting dynamic values. This will be found relevant to the formation of bed ripples (§§4, 5).

(c) *Application to grain flow over a gravity bed*

In the rotating-drum experiments the external stresses were applied wholly at the boundaries, and both concentration and rate of shear were approximately constant from layer to layer. None of these conditions obtain in the case of grain flow over a gravity bed. But the inward stress applied normally at the two boundaries in order to constrain the expansion of the dispersion can clearly be replaced, as regards either boundary, by an equal body force acting on the grains in the same direction, without in any way affecting the stress on the other boundary. Again, the dynamic effects which in the rotating-drum experiments were constant from shear layer to shear layer should be identical with those in any narrow shear layer over a gravity-bed boundary. The only serious doubt concerns the identity of the conditions at the boundary itself or immediately above it. This will be considered in the next section.

Assuming the experimental results are generally applicable, their significance is as follows. In the absence of grain suspension by the eddies of fluid turbulence, if any, the normal grain stress P at the gravity boundary must be equal, under steady conditions, to the normal immersed weight component of all the grains moving over unit bed area, no matter how they are dispersed, and no matter what may be the flow conditions above the boundary.

It follows that from a knowledge of the number G and hence of the dynamic stress ratio T/P at the base of the moving dispersion an equilibrium relation can be defined between the weight or load of moving grains and the resisting shear-stress element T at the bed surface. Further, if, as will also be discussed in the next section, the surface of a loose grain bed can be regarded as a shear plane within the dispersion, so that the normal stress P reacts wholly upon the stationary-bed surface grains, the equilibrium of these under steady conditions can be defined in terms of a static stress ratio or friction coefficient.

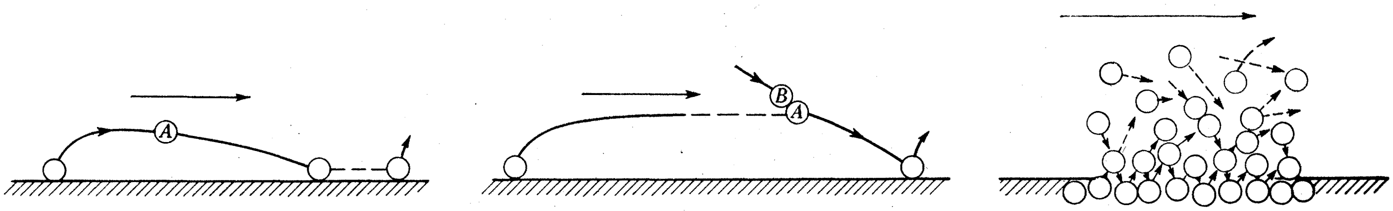
2. Gravity and gravity-free cases compared; saltation

(a) *Saltation in liquids*

When sufficiently few grains are moving for their individual paths to be observed close above a grain bed, i.e. at comparable flow strengths not far above the threshold of first bed disturbance by the fluid, the bed grains, if exceeding a certain size of mass, are seen to jump upwards into the fluid from seemingly random rest positions. The jumping grain subsequently becomes accelerated in the direction of flow, and its path bends over forwards. Gravity ultimately pulls it back to the bed. The path resembles a low arch, as sketched in figure 3*a*. It shows no random deflexions such as would be expected were the influence of turbulent fluid eddies to be effective. Owing to viscous effects in liquids the grain's velocity on return to the bed is insufficient to cause any observable rebound or any disturbance of the bed grains, at any rate at the feeble flow strengths defined above.

This arched and undeflected motion, known as 'saltation', has often been described, e.g. Gilbert (1914), H. A. Einstein (1950), Danel *et al.* (1953). The initial upward acceleration from the bed is accepted as due to a fluid-dynamic 'lift force' (Jeffreys 1929) in combination, probably, with the normal component of an inclined contact force exerted by other bed grains against which it is pressed by a tangential fluid force. Whatever the precise cause, it is very improbable that any upward force continues to act on the grain after its distance from the boundary has exceeded half a diameter or less. Hence most of the observed rise of one to two diameters in the case of liquids should be due to the grain's own inertia, which continues to carry it upwards against gravity.

The same outward jump away from a shear boundary (here reported for the first time) occurs in the gravity-free conditions of the rotating-drum apparatus. It can be seen when the grain concentration and the speed of rotation are both so small that the path of an individual grain can be followed by eye. The jump occurs when a grain A in figure 3b, previously drifting in a constant orbit, gets knocked on to a boundary by a second faster grain B. After the jump, grain A re-attains a constant orbit in which it remains till another chance encounter. The 'height' of the jump was found to increase with the rate of shear, and at the visual limit exceeded 3 diameters.



(a) Gravity case (b) Gravity-free case (c) Gravity case at increased concentrations

FIGURE 3

The rate of shear, of the order of $30\,s^{-1}$, was very large in relation to any comparable threshold rate of shear over a gravity grain bed. For since the grains in this experiment had necessarily the same density as the fluid (water), the threshold rate of shear over a gravity bed would have had to be zero.

Under these conditions such a large excursion as 3 diameters normally through the fluid is to be expected even when the grain has no excess density. For the time taken by viscous fluid drag to develop progressively over the whole grain after a sudden jerk at impact with grain or boundary is comparable with $D^2/4\nu$ (Jeffreys 1929). Assuming that the distance travelled in this time is of the same order as that of unopposed travel at the initial speed during half the time, the initial normal speed away from the boundary required for a travel distance of 3 diameters would be $3D \times 8\nu/D^2$, D being 0·136 cm and ν 0·01 for water, the initial speed becomes 1·76 cm/s. The original orbital speeds of the grains being between 5 and 10 cm/s, this seems easily possible. Had the grains been more dense their inertia would doubtless have carried them further, after the fluid drag had become fully developed. On the other hand, a normal gravity force would have reduced the jump. The initial acceleration away from the boundary could have been due to simple elastic rebound or to a fluid-dynamic lift arising from a sudden retardation by friction at contact with the boundary, or to a combination of both.

Ignoring the random interludes of rest, shown dotted in figure 3a and b, the cycles of events in the two cases are remarkably similar. In both cases the boundary experiences

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a tangential and a normal impulse due to the grain's saltation. In figure 3*a* the tangential impulse is derived from the fluid flow which thereby loses momentum; and the normal impulse is due to gravity. In figure 3*b* both impulses are derived from the opposite boundary via intermediate grains *B*. Since the tangential and normal impulses are necessarily delivered simultaneously to the boundary their ratio is also the stress ratio T/P. Provided it can be assumed that this ratio depends mainly on the relative effects of grain inertia and fluid viscosity, its value should not be seriously affected by the substitution of a normal gravity stress for an externally applied stress at a second boundary.

The main source of possible discrepancy would seem to lie in the nature of the normal impulse delivered to the boundary. If part of the momentum is carried by the fluid surrounding the grain as it strikes the boundary, having been transferred from the grain during its approach, the fluid contribution to the total impulse may be delivered not to the surface grains but to the static fluid between them, whereas in the rotating-drum experiment the effect of the whole impulse was measured together as a stress on the impervious drum wall.

But the mean value at the boundary of any normal stress contribution from the fluid is likely to be small relatively to that from the grains. For the mean tangential grain stress T is derived from a general transfer of tangential momentum to the grains from the body of the fluid. And this transfer is effected by a mean tangential displacement of the grains backward through the fluid. Under steady conditions there can, however, be no corresponding mean normal displacement of the grains relatively to the fluid. So any normal fluid stress on the boundary due to the grain's motion should be residual only, arising from a lack of symmetry in the upward and downward grain velocities. The proportion of this possible fluid contribution to P should diminish with increasing relative grain density, and will evidently be quite negligible in the case of wind-blown grains which exceed 2000 times the density of the fluid.

As the grain concentration over the boundary is increased, observation of individual grain movement becomes impossible. But neither of the simple pictures in figure 3*a* and *b* can remain applicable. As the concentration increases, encounters between moving grains become more frequent, till when $\lambda > 1$ ($C > 8$ or 9 %) they are inevitable, see the lower part of figure 3*c*. Here conditions in the gravity and gravity-free cases should coincide more and more as the concentration is increased.

The delayed development of fluid drag after an encounter now becomes important in both cases. For when λ is large the time between alternate top and bottom encounters on the part of any grain, i.e. between successive reversals of its normal velocity component, should be of the order of $D/2U'$, where U' is the relative layer velocity $\sim D\,\mathrm{d}U/\mathrm{d}y$. And the fluid drag on the oscillatory part of a grain's movement should never become fully developed if

$$U' > 2\nu/D.$$

This gives a critical layer velocity for 1 mm grains in water of only 0·2 cm/s. At higher velocities any doubt as to the possible effects of normal momentum transfers to and from the fluid should tend to disappear.

A further effect of increasing concentration, in the gravity case, should be that an increasing proportion of the fluid's tangential momentum is transferred to the grains above the bed. So the fluid flow at the bed surface should thereby be progressively enfeebled. If so the shear resistance of the substance just above the bed must be increasingly exerted by the grain element in it, and ultimately when a sufficient number of grains are moving $\tau' \to 0$ and $\mathscr{T} \to T$, just as was found in the rotating-drum experiment. But owing to the relatively high density of liquids this change should not be appreciable till the grain population has become large. If and when the change does occur it is evident that the dislodgement of further grains from a grain bed must be by the impact of moving grains and no longer by direct fluid action.

(*b*) *Saltation in air*

With wind-blown grains, on the other hand, a sudden change occurs at the threshold of grain movement (Bagnold 1941). The saltating grains, unlike those in liquids, have a very large relative inertia. The first grain to be dislodged from the bed returns to it with such violence that secondary grains are splashed up into the air stream. These in turn saltate, and a mechanical chain reaction ensues. This results in the number of grains in motion increasing discontinuously from zero. Such a high proportion of the fluid's tangential momentum is thereby transferred to the saltating grains above the bed that the fluid shear resistance τ' at the bed surface becomes negligible and $\mathscr{T} \to T$ at the threshold of first movement.

Hence effects which should develop in liquids only under strong flow are found in air at all flow strengths. As will emerge later, similarity of effect appears to be definable by the criterion G of the preceding section.

(*c*) *Limit to saltation*

At low flow strengths in water, comparably near the threshold of grain movement, it is observed that the typical saltation jumps fade out progressively as the grain size is reduced below a certain limit (see § 5*b*). But the observed diameter limit is considerably greater than the theoretical limit deduced by Jeffreys (1929) from considerations of classical fluid dynamics into which grain density did not enter. Linking this discrepancy with that in the height of rise, 1 to 2 diameters observed (cf. $\frac{1}{3}$ to $\frac{1}{2}$ diameter to which any upward accelerating force should be limited), it appears that the observed fading out of the saltation jump may refer only to the inertial aftermath of the grain's initial upward acceleration.

Under the above flow conditions, the grain paths would be long, as in figure 3*a*, and the delay in the development of fluid drag is likely to be negligible. So the upward inertial travel should be increasingly limited by fluid drag as the drain diameter is reduced.

On this view the observed, as distinct from Jeffreys's theoretical, saltation is a manifestation of the effects of grain inertia *vis-à-vis* those of fluid viscosity. And the reported limits of grain diameter between which the fading out has been observed should be predictable by giving G^2 its values 1500 and 100, which according to figure 2 define the transition from inertial to non-inertial grain shearing. Close agreement will be shown in § 5*b*.

It should be noted that since the dispersive grain stress P has been found by experiment to persist in the absence of any grain inertia effect it is now no longer necessary to assume that when the saltation phenomenon is absent, in the case of fine grains, suspension by fluid turbulence is the only alternative supporting agency.

3. Stress equilibrium in steady flow

(a) General two-dimensional conditions; definitions

(i) *The bed*

The plane surface of a bed composed of loose natural and reasonably uniform grains is taken as the boundary to the flow of both fluid and grains. Any feeble fluid flow through the bed grains is neglected. The bed surface is declined downwards from the horizontal at an angle $+\beta$, so that tangential gravity components are positive in the direction of flow (figure 4). The x axis in the direction of flow, and the horizontal z axis, lie in the plane of the bed surface, and the y axis is normal to it with y measured upwards from the surface. The grain bed extends downwards to a fixed impervious floor at $-y_f$.

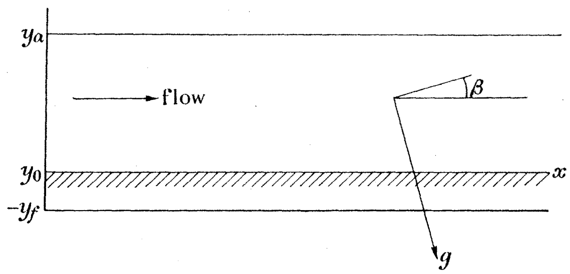

Figure 4. General layout; two-dimensional case of steady flow over grain bed.

The distance y_f is supposed kept at a mean constant value along the flow by the addition of more grains from outside the system, as the impelling forces may be gradually increased and as erosion of the bed would otherwise result. The topmost bed layer therefore remains statistically constituted of the same stationary grains.

(ii) *The unit stress* $\gamma = (\sigma-\rho)\, gD\cos\beta$

The ultimate shear resistance of the bed is that of its topmost layer. If under ideal conditions the whole of this layer were to yield simultaneously, at the plane $y = -D$, the necessary applied tangential stress $\mathscr{T}_0$ would be expected to be given by

$$\mathscr{T}_0 = (\sigma-\rho)\, gD\cos\beta\, C_-\tan\alpha_-, \tag{6}$$

where C_- and $\tan\alpha_-$ are respectively the uniform static values of the grain concentration and of the grain stress ratio within the bed. (The suffixes $_-$ indicate constant static conditions at negative planes $-y$.)

For natural cohesionless grains C_- varies but little, and will be assumed here to be a general constant at 0·63. If $\tan\alpha_-$ is assumed given approximately by the limiting angle of repose it also may be taken as an approximately general constant. For reasonably rounded grains $\tan\alpha_-$ appears to have about the same value of 0·63. The product $C_-\tan\alpha_-$ seems to be still more constant at about 0·4. For though angular grains are often found to stand at a steeper slope than 0·63 their static concentration for random packing is usually less.

The numerical constants in (6) can therefore be dropped, and the stress

$$\gamma = (\sigma-\rho)\, gD\cos\beta \tag{7}$$

emerges as a standard unit of measurement of all relevant stresses. This unit is by no means new (e.g. Shields 1936). But the significance of its physical meaning as implied by (6) seems to have been overlooked. Its importance will become progressively apparent.

(iii) *The applied tangential stress*

The total applied tangential stress $\mathscr{T}_y$, in the same sense as the flow, on any plane $\pm y$ within the grain-occupied zone whether above or below the bed surface, results from the action of the external impelling forces on the system.

The apparent tangential stress $\mathscr{T}_{Fy}$ results from the action of the impelling forces on the fluid alone. And since there is also a tangential gravity component acting on the grains, the total stress is given by

$$\mathscr{T}_y = \mathscr{T}_{Fy} + (\sigma - \rho)\, g \sin\beta \int_y^{\infty} C\,\mathrm{d}y, \tag{8}$$

where σ and ρ are the constant densities of grains and fluid respectively.

In the case of wholly parallel flow $\mathscr{T}_{Fy}$ is given by

$$\mathscr{T}_{Fy} = (y_a - y)\left(\rho g \sin\beta \pm \frac{\mathrm{d}p}{\mathrm{d}x}\right), \tag{9}$$

where y_a (figure 4) defines some upper parallel plane above the grain dispersion where the rate of shear is zero, e.g. the free surface of a liquid, or some plane below the roof of a closed duct. The pressure gradient $\mathrm{d}p/\mathrm{d}x$ enters only in the case of a closed duct; and in the present two-dimensional treatment (9) is only applicable to a very broad flat duct.

Dividing by the unit stress γ, and at the same time converting y into units of grain diameter, so that $y = hD$, the characteristic applied tangential stress Θ_h becomes

$$\Theta_h = \mathscr{T}_h/\gamma = \Theta_{Fh} + \tan\beta \int_y^{\infty} C\,\mathrm{d}h, \tag{10}$$

or for wholly parallel flow

$$\Theta_h = \tan\beta\left\{\frac{\rho}{\sigma-\rho}(h_a - h) + \int_h^{\infty} C\,\mathrm{d}h\right\} \pm \frac{D}{\gamma}\frac{\mathrm{d}p}{\mathrm{d}x}(h_a - h)\cos\beta. \tag{10a}$$

The tangential stress $\mathscr{T}_0$ or its dimensionless measure Θ_0 at the bed surface appears to be the natural, and indeed the only practicable quantity by which to define the strength of the flow for purposes of any general theory concerning grain movement by the flow. The mean apparent applied stress along the flow can be estimated under practical conditions where the relative influence of side boundaries is reasonably small, e.g. flow in shallow open channels. And in most such cases the value of the integral in (8) and (10) is so small that the extra grain contribution can be neglected provided the mean flow over a considerable length of the channel can be regarded as parallel. But as will appear in §§ 4*b* and 13 this contribution becomes important in connexion with the spontaneous formation of local bed features. For the bed inclination β may then develop large variations along the flow.

Unfortunately, the tangential stress on the bed surface cannot as yet be measured or estimated in the case of grain flow through a circular pipe.

(iv) *Bed load and suspended load*

A load as generally understood is a normal gravity stress, and the word will be used here in this sense. The same word is, however, still sometimes applied to the time rate at which the load of transported grains is carried along a stream. This quite different quantity will here be called the 'transport rate'.

In the present context the load over any plane h ($h = y/D$) is defined as the normal immersed weight component of all the grains above that plane, per unit bed area. It is expressed by

$$\text{load} = (\sigma-\rho)\, g \cos\beta \int_y^\infty C\,\mathrm{d}y = (\sigma-\rho)\, gD \cos\beta \int_h^\infty C\,\mathrm{d}h$$
$$= \gamma \int_h C\,\mathrm{d}h. \tag{11}$$

The conformable characteristic load, denoted by χ, is therefore

$$\chi = \frac{\text{load}}{(\sigma-\rho)\, gD\cos\beta} = \int_h^\infty C\,\mathrm{d}h. \tag{11 a}$$

χ has the simple meaning that the grain content of the load is χ times that of one grain layer of the stationary bed: $\int_{h=-1}^{0} C\,\mathrm{d}h = C_-$.

The absolute mass of the load χ is $\chi\sigma D$.

When part of the load is supported in suspension by the eddies of fluid turbulence, this part cannot also be supported by the normal grain stress P due to the shearing of the whole dispersion. In general, therefore, the load must be assumed to consist of two parts.

The bed load is defined as that part of the load whose normal immersed weight component is in normal equilibrium with the grain stress P. This stress is transmitted downwards via the dispersed grains to the stationary grains of the bed upon which it therefore ultimately rests.

The suspended load is defined as that part of the load whose like weight component is in equilibrium with a normal fluid stress originating in impulses by turbulent eddies. This stress is supposed transmitted not to the bed grains but between them as an excess static fluid pressure. This excess pressure over the pressure due to an equal column of grainless fluid has recently been measured (Bagnold 1955).

Neither the absolute nor relative distributions of the two load elements with distance from the bed are known. So it must be assumed that the grain content C of every unit volume may consist of elements C_b and C_s attributable respectively to bed and suspended load. We therefore have, in dimensionless terms,

$$\left.\begin{aligned} \text{bed load } \chi_b &= \int C_b\,\mathrm{d}h = P/\gamma, \\ \text{suspended load } \chi_s &= \int C_s\,\mathrm{d}h, \\ \text{whole load } \chi &= \chi_b + \chi_s. \end{aligned}\right\} \tag{12}$$

The normal dynamic grain stress P is clearly additive to the normal static grain stress within the bed. So, irrespective of whether or not a suspended load is present, the normal stress between the bed grain layers at any depth $-h$ is represented by

$$\chi_{b\,(\text{static})} = \int_0^\infty C_b\,\mathrm{d}h + C_- h_-. \tag{13}$$

And as far as normal stresses are concerned the moving bed-load grains may be deemed to constitute a part of the bed which happens to be dispersed above it.

(*b*) *Shear stresses*

For equilibrium under steady flow the applied tangential stress Θ_h at any plane $\pm h$ must be resisted by an equal internal stress.

Within the moving grain dispersion the internal resisting stress is exerted partly by the grains and partly by the intergranular fluid. By (1) and (10)

$$\Theta = \Theta_F + \tan\beta\int_h^\infty C\,\mathrm{d}h = \frac{T}{\gamma}+\frac{\tau'}{\gamma}. \tag{14}$$

But $T = P\tan\alpha = \gamma\chi_b\tan\alpha$ by (2) and (12), whence

$$\chi_b\tan\alpha = \Theta - \tau'/\gamma. \tag{15}$$

Or in terms of apparent applied stress, $\Theta_F = \Theta - \tan\beta(\chi_b+\chi_s)$,

$$\chi_b(\tan\alpha - \tan\beta) = \Theta_F - \frac{\tau'}{\gamma} + \tan\beta\,\chi_s. \tag{16}$$

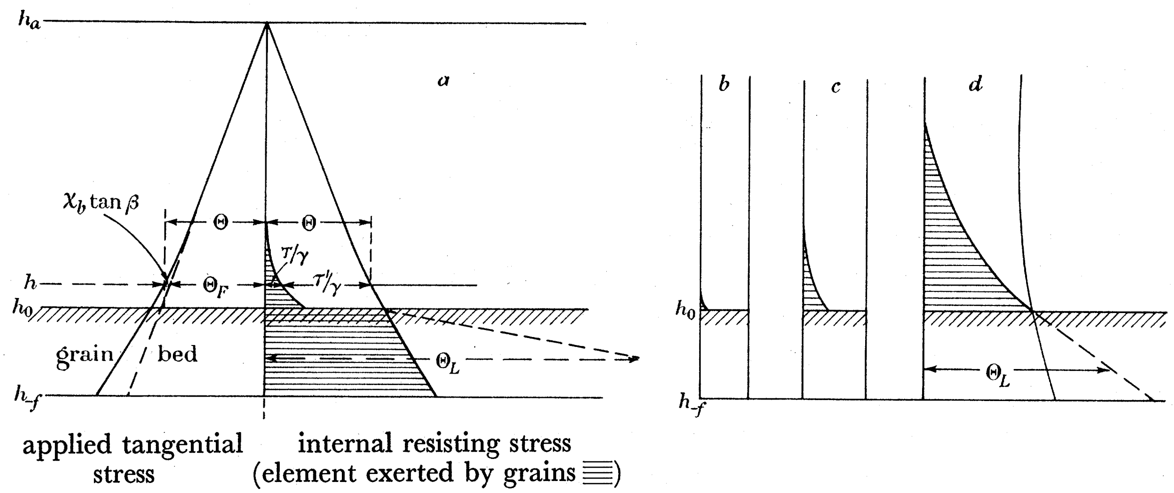

FIGURE 5. (*a*) Constitution of characteristic shear stress. (*b*) to (*d*) Internal resisting stress of (*a*) with h_a very large; showing progressive increase in the ratio T/τ as the total tangential stress is increased over a grain bed.

The value of τ' in the usual case of a turbulent fluid is unknown; but for two-dimensional parallel flow the constitutions of the opposing applied and resisting stresses which together comprise the shear stress at any plane should be somewhat as sketched in figure 5*a*. The progressive increase in the ratio T/τ' with increase in the total shear stress is sketched in figure 5*b*, *c* and *d*.

Below the bed surface τ' is zero by definition, and $\tan\alpha$ is constant at $\tan\alpha_-$. But static friction introduces an indeterminacy; and the ultimate yield stress Θ_L of the bed (excluding the topmost layer) will generally exceed Θ_-. From (13) and (15)

$$\Theta_L = \chi_{\text{static}}\tan\alpha_- = (\chi_{b0}+C_-h_-)\tan\alpha_- \tag{17}$$

and

$$-\frac{\partial\Theta_L}{\partial h} = C_-\tan\alpha_- \sim 0{\cdot}4 \quad (\S\,3\,a\,(\text{ii})).$$

But from (10)

$$-\frac{\partial\Theta}{\partial h} = \frac{\partial\Theta_F}{\partial h} + \tan\beta\,C_-.$$

From figure 5 it will be seen that the condition that the bed shall shear only at its surface is that $\partial\Theta/\partial h < \partial\Theta_L/\partial h$ or

$$C_{-} \tan\alpha_{-} \sim 0{\cdot}4 > \frac{\partial\Theta_F}{\partial h} + \tan\beta\, C_{-}, \tag{18}$$

which for wholly parallel flow in open channels where $\mathrm{d}p/\mathrm{d}x$ is zero becomes, from (10*a*),

$$C_{-} \tan\alpha_{-} \sim 0{\cdot}4 > \tan\beta \left\{\frac{\rho}{\sigma-\rho} + C_{-}\right\}. \tag{18a}$$

In most, but not all, practical cases $\rho/(\sigma-\rho)$ is of the same order as or smaller than unity and $\tan\beta$ is very small; so condition (18*a*) will be satisfied. What happens when it is not satisfied will be dealt with in § 12*b*.

(*c*) *Equilibrium at the surface of a plane grain bed*

Under the ideal conditions of purely laminar fluid flow over a bed of uniform spheres perfectly piled and therefore equally exposed, no bed grains should move (provided (18) is satisfied) until the tangential fluid stress Θ_0 on the bed boundary is raised to the value $C_{-} \tan\alpha_{-}$ (§ 3*a* (ii)). The whole topmost layer should then be peeled off simultaneously and become dispersed.

If this dispersion were to be wholly removed as a suspension, successive topmost bed layers would also be peeled off. Hence even under ideal conditions no grain bed could persist at all when Θ_0 exceeded the above critical value of about 0·4. The persistence, in fact, of a grain bed at applied tangential stresses more than 12 times this value (Bagnold 1955) demonstrates the necessity for a bed load χ_{b0} which creates an additional resisting stress at the bed surface.

The surface grains of a natural bed are not uniformly exposed. Nor under a turbulent fluid is the applied tangential stress steady. As a result individual grains begin to be moved at a lower value of Θ_0.

Let $\Theta_{(t)}$ represent the conventional threshold value of Θ_0, defined as that needed to keep in motion the minimum observable number of grains over a bed surface which is plane on a scale larger than that of a few grain diameters. (Another threshold value Θ_t differently defined will be introduced in the next section.) $\Theta_{(t)}$ has been found from experiment to depend partly on the Reynolds number $(D/\nu)\sqrt{(\tau_0/\rho)}$ (Shields 1936) and partly on the state of general turbulence of the whole fluid (White 1940).

Experimental values of $\Theta_{(t)}$ under liquid flow range from 0·033 to 0·06 for fully turbulent flow to 0·2 (exceptionally 0·26) for wholly laminar flow. Under a wind a lower value of $\sim$0·01 (Bagnold 1936; Zingg 1950) can be accounted for by a great increase in the probability of observing first grain movement introduced by the chain reaction effect (§ 2*b*).

Under liquid flow the bed surface grains remain undisturbed by moving grains above them (§ 2*a*). Assuming the bed to be randomly piled (natural grains) and homogeneous over an area large compared with that of a single grain, it may be supposed that its surface contains:

(i) always some grains dislodgeable by the fluid at the threshold stress $\Theta_{(t)}$ but none at any lesser stress;

(ii) no individual grains which, by reason of their own weight and surroundings only, have an ultimate shear strength capable of withstanding a fluid force equivalent to a distributed tangential stress exceeding $\Theta_0 = C_- \tan\alpha_- \sim 0{\cdot}4$.

When Θ_0 is raised to some constant value $> \Theta_{(t)}$, grains will by assumption (i) go on being eroded from the bed until the extra resisting stress $T_0/\gamma = \chi_{b0} \tan\alpha_0$ at the bed surface, due to the load χ_{b0} of dispersed eroded grains, just offsets the applied stress increase $\Theta_0 - \Theta_{(t)}$. A steady state is then reached.

Equation (15) becomes at the bed surface

$$\chi_{b0} \tan\alpha_0 = \Theta_0 - \frac{\tau_0'}{\gamma}, \tag{19}$$

where $\tan\alpha_0$ is the dynamic stress ratio to be determined by the number G_0 just over the bed surface. From this equilibrium relation, in conjunction with assumption (i), it follows that as Θ_0 is increased the stress element τ_0'/γ cannot vary from its threshold value $\tau_0/\gamma = \Theta_{(t)}$ so long as the concentration λ_0 just over the bed is not so large as seriously to affect the internal shear resistance of the intergranular fluid, i.e. so long as $\tau_0' \sim \tau_0$.

Over an appreciable lower range of Θ_0 therefore (15) and (19) can, for a plane grain bed, be written as

$$\chi_{b0} \tan\alpha_0 = \Theta_0 - \Theta_{(t)}, \tag{20}$$

and the bed load becomes determinate in terms of Θ_0 and the experimental parameter $\Theta_{(t)}$, both of which may be supposed known.

It also appears to follow that the residual applied tangential stress $\tau_0'/\gamma = \Theta_0 - \chi_{b0} \tan\alpha_0$ which is not resisted by the grain stress T of the bed load must be in equilibrium with the residual inherent static resistance still exerted by some of the bed grains by reason of their own weight. But by assumption (ii) this residual resistance must disappear when Θ_0 exceeds $C_- \tan\alpha_-$. So τ_0'/γ should then also disappear. We also have the independent inference from the rotating-drum experiment that in turbulent fluids τ' must in any case dwindle to an extremely small relative value as the concentration λ approaches its 'fluid' limit of 14, and as the turbulence is suppressed.

These two inferences together suggest that as $\Theta_0 \to C_- \tan\alpha_- \sim 0{\cdot}4$ in the case of liquids (20) can no longer hold because τ_0'/γ dwindles to zero from its previous constant value $\Theta_{(t)}$; and that when $\Theta_0 > 0{\cdot}4$, (20) should become

$$\chi_{b0} \tan\alpha_0 = \Theta_0. \tag{20a}$$

In the case of wind-blown grains, however, the same condition obtains at the threshold of grain movement (§ 2*b*) because the agitation of the whole bed surface by the 'ballistic' type of saltation destroys all the inherent shear strength of the bed grains as soon as the chain reaction is set off by first-moved grain (cf. when agitated, a heap of dry sand flows out flat).

So according to the above considerations the bed-load flow of grains under a turbulent water stream should become quantitatively similar to that of wind-blown sand when Θ_0 in the former case exceeds about 0·4. This is strikingly confirmed by the experimental evidence given in §§ 8 and 9.

Curves of (20) and (20*a*) are plotted in figure 6 on double logarithmic scales for three representative cases. Curve (*a*) of (20*a*) is for wind-blown grains; $\Theta_{(t)}$ becomes zero at first movement. Curve (*b*) is for turbulent liquid flow over a plane bed surface; $\Theta_{(t)}$ has been given Shields's minimum value of 0·033 up to $\Theta_0 = 0{\cdot}2$, and has thereafter been diminished progressively to zero at $\Theta_0 = 0{\cdot}4$. (At $\Theta_0 = 0{\cdot}4$ the resulting increase in χ amounts to only 8 %.) Curve (*c*) will be referred to shortly. Curve (*d*) is for a constant $\Theta_{(t)}$ of 0·2 in the case of wholly laminar fluid flow over natural grains.

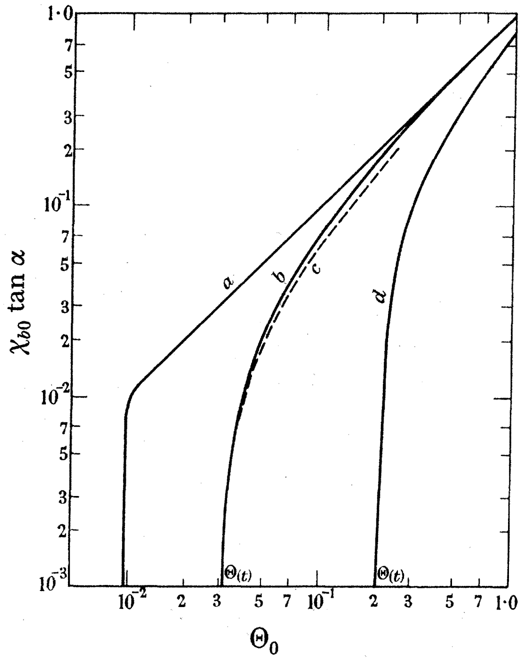

FIGURE 6. Curves of $\chi_{b0} \tan\alpha = \Theta_0 - \tau_0/\gamma$. (*a*) Wind-blown grains: τ_0' vanishes when first bed grain is moved. Liquid-driven grains: (*b*) turbulent flow, stable bed; τ_0'/γ constant at $\Theta_{(t)}$ till concentration C_0 begins to suppress turbulence at bed surface, say at $\Theta_0 = 0{\cdot}2$, then dwindles to zero at $\Theta_0 = 0{\cdot}4$. (*c*) Turbulent flow; maximum effect of primary form-drag;

$$\chi_{b0} \tan\alpha_0 = 0{\cdot}86(\Theta_0 - \Theta_{(t)}).$$

(*d*) Viscous flow (hypothetical); τ_0'/γ should not diminish from its initial value $\Theta_{(t)}$.

4. Small-scale bed features under a turbulent liquid current; form-drag

(*a*) *Deficiency of static bed surface shear resistance*

Equation (19) or its approximation (20) for low values of Θ_0 in liquids defines the dynamic conditions at the base of the moving grain dispersion which are necessary if a steady load is to be maintained. It implies that when under turbulent liquids Θ_0 is less than its critical value $C_-\tan\alpha_- \sim 0{\cdot}4$ over a plane grain bed some inherently resistant bed grains on the surface must be still capable by themselves of withstanding the residual applied tangential stress $\tau_0'/\gamma = \Theta_0 - \chi_{b0}\tan\alpha_0$ not already opposed by the bed-load resistance.

This will be clear from fig. 7 *a*. The necessary static resistance to τ_0'/γ is defined as being due to a certain static grain load $\delta\chi$. This consists in reality of certain still resistant grains scattered over the exposed surface; but it is here idealized as a thin statistical solid layer lying between the surface h_0 and a plane $-h'$ just below ($h < 1$ diam.).

So let

$$\tau_0'/\gamma = \delta\chi \tan\alpha_-. \tag{21}$$

The ultimate shear resistance of the bed at the plane $-h'$ will be $(\chi_{b0}+\delta\chi)\tan\alpha_-$.

The applied tangential stress $\Theta_{-h'}$ at the plane $-h'$ may be taken as $\Theta_0+\delta\chi\tan\beta$; for provided the whole flow depth $\gg h'$ the increment of the fluid contribution Θ_F can be neglected. Whence by (19)

$$\Theta_{-h'} = \chi_{b0}\tan\alpha_0+\frac{\tau_0'}{\gamma}+\delta\chi\tan\beta.$$

The criterion therefore that no bed grains are moved once (19) is satisfied under constant applied stress is

$$(\chi_{b0}+\delta\chi)\tan\alpha_- > \chi_{b0}\tan\alpha_0+\frac{\tau_0'}{\gamma}+\delta\chi\tan\beta.$$

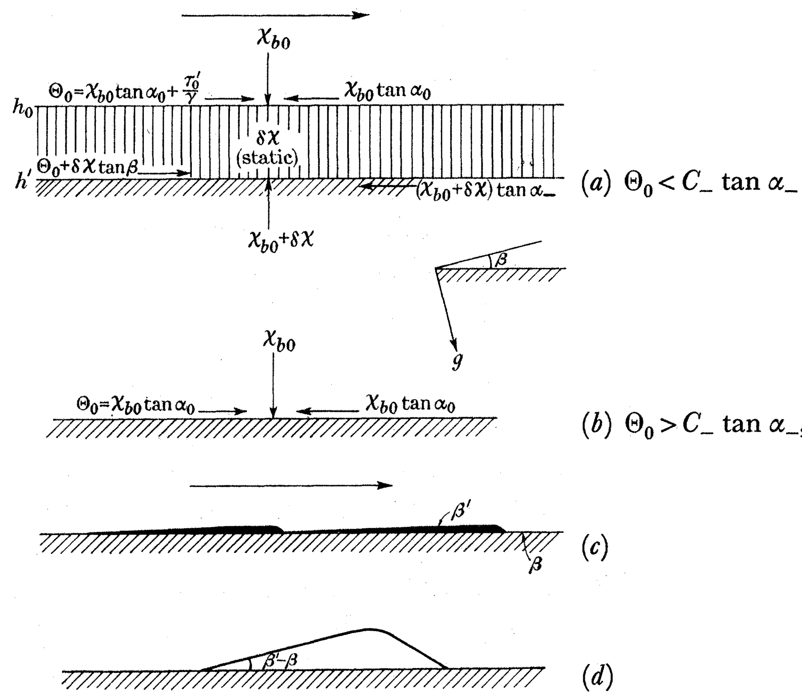

FIGURE 7

Substituting for $\delta\chi$ by (21) this becomes

$$\chi_{b0}\tan\alpha_- > \chi_{b0}\tan\alpha_0+\frac{\tau_0'}{\gamma}\frac{\tan\beta}{\tan\alpha_-}, \tag{22}$$

where τ_0'/γ can be replaced by the constant parameter $\Theta_{(t)}$ for the lower range of Θ_0.

When (22) is not satisfied there will be a deficit in the resistance which any plane grain bed can exert, of magnitude

$$\frac{\tau_0''}{\gamma} = \chi_{b0}(\tan\alpha_0-\tan\alpha_-)+\frac{\tau_0'}{\gamma}\frac{\tan\beta}{\tan\alpha_-}. \tag{23}$$

Bed grains will therefore go on being eroded. Since these cannot be supported as a bed load without an increase in the applied stress they must be re-deposited on the bed. Conditions at a plane bed surface become unstable.

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An ultimate steady state can, however, be attained if the eroded grains are re-deposited in such a way that some new and additional tangential resistance is created. Provided the fluid is turbulent or potentially so, it seems inevitable that the bed surface must pucker into raised features capable of exerting a body resistance or 'form-drag' transmitted to the fluid by comparatively large-scale vorticity created in the hollows between them. If no other complications intervene the cross-section of the bed features should become so adjusted that their additional form-drag just makes good the deficit τ_0''/γ in the granular resistance.

The deficit τ_0''/γ in the surface granular resistance should, however, disappear when Θ_0 is increased to exceed about 0·4. And the bed features due to it should also disappear. For if as was suggested in the preceding subsection τ_0'/γ virtually vanishes from an otherwise turbulent liquid when Θ_0 approaches 0·4, at which the last remnants of inherent static granular resistance vanish, the statistical bed layer of figure 7 *a* becomes non-existent (see figure 7 *b*).

It is significant that in all the relevent literature on grain flow in laboratory water channels bed features are in fact reported to disappear just when Θ_0 approaches 0·4 (§ 9 *d*).

(*b*) *Primary ripple features and primary form-drag*

Consider the implications of (23) at the more commonly experienced lower values of Θ_0. Zero τ_0''/γ should define the critical conditions under which an initially plane grain bed should cease to persist; and the positive value of τ_0''/γ should be a measure of the relative prominence of the features.

Experimental data about bed features formed under a liquid current derive mostly from laboratory work with open water channels, in which the downward slope $+\tan\beta$ of the bed is of the order of 10^{-2} to 10^{-3}. Giving $\tau_0'/\gamma \sim \Theta_{(t)}$ a representative value of 0·04 for turbulent liquids (§ 3 *c*), the last term in (23) would range between say 6×10^{-4} and 6×10^{-5}.

No direct quantitative data are available as to the magnitude of grain loads moving over grain beds. But since χ_0 is simply the ratio (grain content of load)/(content of one bed grain layer) (§ 3 *a* (iv)), an easy estimate of it can be made by counting the number of grains seen or photographed at any instant moving over a bed area containing any convenient number of bed-surface grains. In the region of the threshold of movement, to which much experimental work has been confined, one moving grain to 1000 bed grains, or $\chi_0 = 10^{-3}$, would be quite appreciable. On the other hand, at $\Theta_0 = 0{\cdot}4$, χ_{b0} is by (20 *a*) of the order of unity.

If figure 2 is assumed applicable, at any rate for reasonably rounded natural grains, the dynamic stress ratio $\tan\alpha_0$ just above the bed ranges between 0·32 for large, inertial grains, and 0·75 in the viscous region. And if for the same type of grain the static stress ratio $\tan\alpha_-$ just below the bed surface is taken as 0·63 (angle of repose ~32°), the difference $(\tan\alpha_0 - \tan\alpha_-)$ ranges between about $-0{\cdot}31$ and $+0{\cdot}12$, with zero at about $G^2 = 440$.†

It seems from the above that:

(i) Sufficiently close to the threshold of movement, primary bed features should always develop (given adequate time), however large the grains, on any bed surface having a finite downward (positive) slope.

† Angular grains are known to give larger values of tan α_-. But it seems probable that if experimental difficulties allow the rotating drum work to be repeated with such grains, the same angularity would be found to cause a general increase in the dynamic values also, leaving the three critical values 100, 440 and 1500 of G not very different. See the test results given in the next section.

(ii) If $\tan\alpha_- > \tan\alpha_0$ ($G^2 > 440$, large grains), features which may develop close to the threshold of movement should disappear as Θ_0, and with it the load χ_{b0} is increased. The persistence of the features should then depend on downward slope $+\tan\beta$.

(iii) If $\tan\alpha_- < \tan\alpha_0$ (small grains), the features should get more prominent with increasing applied stress until either some other effect intervenes or $\Theta_0 \to 0{\cdot}4$. But this should not apply when the grains are very fine and exhibit mutual cohesion. For in this case $\tan\alpha_-$ is likely to be so great that plane bed surfaces are always stable.

(iv) Since the role of these primary features is supposed to be solely to exert a resisting form-drag by eddy formation, it might be assumed that the drag tends to be located in the neighbourhood of the individual lee faces rather than on the upstream faces. If so, as the features mature and the local upstream slopes $\tan\beta'$ diminish or become negative, the local resistance deficit on these faces would tend towards zero. This suggests that the local upstream slopes $\tan\beta'$ should be roughly limited to

$$\tan\beta' = -\chi_{b0}(\tan\alpha_0 - \tan\alpha_-)\frac{\tan\alpha_-}{\Theta_{(t)}}, \tag{24}$$

obtained by putting $\tau_0''/\gamma = 0$ in (23). As a result the upstream faces would tend to become plane surfaces. For example:

Large grains. Let $\chi_{b0} = 5\times10^{-4}$ (small but observable),

$$\tan\alpha_0 - \tan\alpha_- = -0{\cdot}31, \quad \Theta_{(t)} = 0{\cdot}06, \quad \tan\beta_{\text{mean}} = 10^{-2}.$$

Then $\tan\beta' = 1{\cdot}75\times10^{-3}$ (still downward) and $\beta - \beta' \sim +\frac{1}{2}^0$ only (see figure 7*c*).

Small grains. Let $\chi_{b0} = 0{\cdot}2$, $(\tan\alpha_0 - \tan\alpha_-) = +0{\cdot}12$, $\Theta_{(t)} = 0{\cdot}06$, $\tan\beta_{\text{mean}} = 10^{-3}$. Then $\beta' \sim -14^0$ (upward) (see figure 7*d*).

The inferential picture begins to outline both the conditions for, and the appearance of, the small-scale features or 'bed ripples' actually observed under steady water currents.

Provided no secondary effects intervene, (19) for the steady state becomes

$$\chi_{b0}\tan\alpha_0 = \Theta_0 - \frac{\tau_0'}{\gamma} - \frac{\tau_0''}{\gamma}. \tag{25}$$

Substituting for τ_0''/γ by (23)

$$\chi_{b0}\tan\alpha_0 = a\left\{\Theta_0 - \frac{\tau_0'}{\gamma}\left(1 + \frac{\tan\beta}{\tan\alpha_-}\right)\right\}$$

or

$$= a(\Theta_0 - \Theta_{(t)}) \text{ approx.,} \tag{26}$$

where $a = \tan\alpha_0/(2\tan\alpha_0 - \tan\alpha_-)$ is unity whenever $\tan\alpha_0 < \tan\alpha_-$, and has a minimum at about $+0{\cdot}86$ when for small grains $\tan\alpha_0 - \tan\alpha_- = -0{\cdot}12$.

The maximum effect of the primary form-drag on the bed load χ_{b0} is small. The growth of this effect with increasing Θ_0 is sketched by the broken curve (*c*) of figure 6.

(*c*) *Secondary ripple features and secondary form-drag; the parameter* Θ_t

When the mean bed slope $\tan\beta$ is negligible, both the primary form-drag τ_0''/γ and the up-tilt of the ripple faces are negligible at the threshold $\Theta_{(t)}$ of movement. But both increase with progressive increase in Θ_0, and the necessary vorticity in the ripple hollows becomes stronger.

A first and minor secondary effect of these increases should be a general rounding-off of the ripple cross-section. But at some critical value of τ_0''/γ a sudden and radical change should take place. For with increase in the up-tilt of the ripple faces, the fluid flow close over the alternate down and up faces must become appreciably diverging and converging; and its turbulence should therefore be alternately stimulated over the hollows and suppressed over the up-slopes.

Once scouring occurs in the hollows, these will tend to deepen. The debris will tend to be deposited on the up-slopes, which will therefore get steeper. So the result should augment the cause, and the primary ripple system should become suddenly unstable. Fluid vorticity should be greatly increased. Hence whatever ultimate relation limits the size of the secondary features, the form-drag created by the new ripple system should exceed the value actually needed for the steady state.

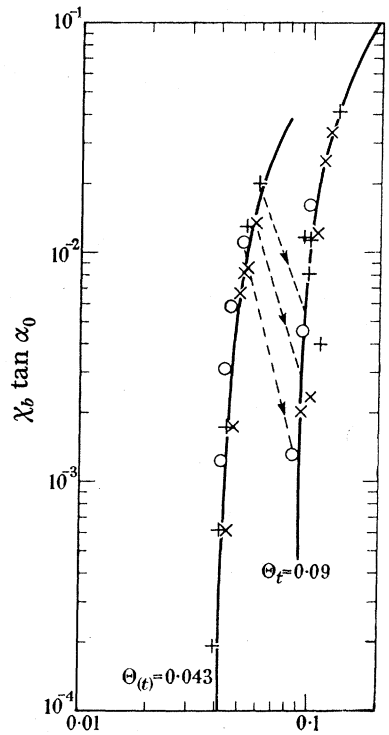


FIGURE 8. Bed loads, reduced from relevant experimental transport rates of plot III, figure 14, showing clear-cut discontinuity consequent to onset of secondary instability. Symbols ○, ×, + refer to different constant bed-slope settings and to relative flow depths decreasing in this order. Dashed arrows indicate respective sequences of bed-load values as applied stress is increased progressively.

Accordingly, an experimental curve of χ_{b0} against Θ_0 such as that sketched in figure 6 should conform to curve (c) (insignificant primary form-drag) from the threshold up to some critical value of χ_{b0}, but should then show an abrupt discontinuity in the value of Θ_0 necessary for a continuous increase in χ_{b0}. This discontinuity should occur when the vorticity in the hollows gets strong enough to cause scour.

No direct experimental values for the bed load being available, confirmation must be sought in the existing data on its transport rate which alone is readily measurable. The transport rate, as will be shown in §7, part II, is proportional for any given system to $\chi_{b0}\Theta_0^{\frac{1}{2}}$. Figure 8 shows the relevant experimental data for quartz grains in water as given in plot III, figure 14, but with the ordinates adjusted in scale by a factor proportional to $\Theta_0^{-\frac{1}{2}}$. This plot is representative of a number of others (figures 13 to 15) for varying grain sizes below a certain limit. The predicted discontinuity appears consistently. It has not been previously explained.

Moreover, since $\chi_{b0} \sim \dfrac{\tau_0''/\gamma}{\tan\alpha_0 - \tan\alpha_-}$ by (23), the critical value of χ_{b0} at which the primary ripple system breaks down should increase, as, with increase in grain size, $\tan\alpha_0 \to \tan\alpha_-$, till the breakdown fails to occur at all, and the primary system persists. This is also borne out by the experimental data (see figures 13 to 15).

The two continuous curves superimposed on figure 8 are those given by $\chi_{b0}\tan\alpha_0 = \Theta_0 - c$, where c has been taken in each case from the experimental plot as the value of the vertical asymptote Θ_0 corresponding to zero χ_{b0}. For the initial, left-hand branch, $c = \Theta_{(t)}$ as in figure 6. But it will be seen that both the calculated curves fit the rest of the experimental data equally well. So it is proposed to introduce a more general parameter Θ_t, definable at will either as the conventional plane-bed threshold stress $\Theta_{(t)}$ or as the secondary 'threshold' stress in relevant cases. Equation (20) therefore becomes

$$\chi_{b0}\tan\alpha_0 = \Theta_0 - \Theta_t. \tag{27}$$

This secondary threshold Θ_t seems to need more experimental attention. An analysis of the limited evidence referred to above (U.S. Waterways 1935) suggests that over the range of nearly 2 to 1 in the water-flow depths Θ_t remains sensibly constant for any given grain size. But the critical load at which the primary system breaks down, and at which $\Theta_{(t)}$ ceases to be relevant, appears to decrease significantly with increase in flow depth. This seems reasonable in view of the increasing scale and influence of the general turbulence. So at flow depths exceeding the few centimetres of the U.S.W. experiments $\Theta_{(t)}$ may be of little practical importance where plane bed surfaces fail to persist.

(*d*) *Bed features in general*

The spontaneous deformation of a plane grain bed into any pattern of raised features may be supposed to be due to some kind of instability. But since a number of quantities are involved, a number of different mutually unstable conditions may arise between them.

The features just dealt with, and usually observed on a small scale only, appear to result, primarily from an effect somewhat analogous to the crumpling of insufficiently loaded paper by the use of a rubber, and at a subsequent stage from an unstable interaction between local fluid turbulence and local fluctuations in the bed angle. These can only occur in a turbulent fluid at low values of G when (i) dynamic friction above the bed surface interface exceeds static friction below, (ii) moving grains above have insufficient inertia to disturb bed grains below, and (iii) the fluid stress applied at the interface must consequently remain appreciable.

But at the very high values of G obtaining when ordinary sands are wind-blown none of these conditions is satisfied. The wind-blown sand ripples, of distinctive pattern and occurring wholly under super-inertial conditions, are therefore very unlikely to have any counterpart under liquids. Under like conditions grain beds under liquids would be plane and unrippled. The wind-blown ripples have been attributed (Bagnold 1941) to a 'ballistic' interaction between the effects of a bombardment of the bed surface and the local incidence angle of the bombarded surface.

There remains a class of bed features commonly called 'dunes' which can occur in nature on a very large scale, both in air and in water. The theory seems capable of accounting for these as resulting from an unstable interaction between fluctuations, along the flow, of the gravity slope of the bed and of the transport rate of the grains over it (§13).

5. Evaluation of the grain-flow number, G; predicted and observed conditions for saltation and for bed rippling

(a) *Evaluation of* $G_0 = \frac{D}{\eta}\sqrt{\frac{\sigma T_0}{\lambda_0}}$ *over a grain bed*

Neither of the variables T_0 and λ_0 at the base of the sheared grain dispersion are directly measurable. So their ratio must somehow be inferred. From the experimental evidence (§1*b*) τ_0' dwindles to insignificance compared with T_0 as the concentration λ_0 approaches its constant 'fluid' limit of about 14. So it will be assumed (with greater certainty in the case of a turbulent fluid) that $\gamma\Theta_0 \to T_0$ as $\lambda_0 \to 14$. And from the reasoning of §3*c*, τ_0' disappears at the threshold of grain movement in the case of 'ballistic' systems, e.g. wind-blown sand; and at $\Theta_0 = C_- \tan\alpha_- \sim 0{\cdot}4$ in that of non-ballistic, liquid systems. We have, therefore,

$$\left.\begin{array}{r}\text{'ballistic' systems, all values of } \Theta_0\\ \text{liquids, } \Theta_0 > 0{\cdot}4\end{array}\right\} \quad \left.\begin{array}{l} G_0^2 \sim \dfrac{\sigma D^2 \gamma \Theta_0}{14\eta^2} = \dfrac{D^3\sigma(\sigma-\rho)\,g\cos\beta\,\Theta_0}{14\eta^2},\\ \text{or}\quad D^3 \sim \dfrac{14\eta^2 G_0^2}{\sigma(\sigma-\rho)\,g\cos\beta\,\Theta_0}.\end{array}\right\} \tag{28}$$

This leaves only the liquid case when $\Theta_0 < 0{\cdot}4$ and the fluid stress element τ_0'/γ persists. At the critical stress $\Theta_0 = 0{\cdot}4$, T_0/λ_0 is by the above assumption equal to $0{\cdot}4\gamma/14 = 0{\cdot}029\gamma$. In this case T_0 gives place progressively to τ_0' as Θ_0 is reduced. When regarded as distributed quantities per unit bed area, both T_0 and λ_0 tend towards zero together as the number of moving grains approaches zero at the threshold of motion. But if T_0 is regarded as the sum of all the tangential resistances exerted by the individual moving grains, and λ_0 as the effective free distance of each grain from the bed, then both quantities remain finite for the movement of a single grain. It is possible therefore that the ratio T_0/λ_0, and hence G, remains sensibly constant as Θ_0 is reduced from 0·4 to the threshold. Experience goes to support this, provided the observed saltation phenomenon is a grain inertia effect as suggested in §2. For if G_0 were to vary appreciably over the observable, say, fivefold range of Θ_0 from the threshold upwards, some mention would be found in the literature of an apparent connexion between Θ_0 or the flow strength and the observed onset of saltation. But no such mention can be traced, from Gilbert (1914) onwards. Assuming therefore that G_0 remains sensibly constant we have:

$$\left.\begin{array}{l}\text{liquids, } \Theta_0 < 0{\cdot}4 \quad G_0^2 \sim \dfrac{0{\cdot}029\gamma\sigma D^2}{\eta^2} = \dfrac{0{\cdot}029 D^3\sigma(\sigma-\rho)\,g\cos\beta}{\eta^2},\\ \text{or} \qquad\qquad D^3 \sim \dfrac{35\eta^2 G_0^2}{\sigma(\sigma-\rho)\,g\cos\beta}.\end{array}\right\} \tag{29}$$

Relations (28) and (29) are rough estimates only, and might be in error by $\pm 50\%$. But, being based entirely on the critical values 14 of λ and $C_- \tan \alpha_-$ of Θ_0, they are free of any arbitrary numerical assumption.

(b) *Diameter limits between which saltation fades out*

Durand (1952)—statement clarified in subsequent correspondence—found that as the diameter of quartz grains under a water current is reduced, the saltation phenomenon faded out progressively between the diameters shown below. Owing to the uncertainty of estimation from the appearance of the grain paths only, the stated diameter range is put conservatively wide.

No corresponding quantitative figures are available for wind-blown grains because of the difficulty of tracing the paths of the necessarily very small grains. But from common experience 'sand' which saltates violently becomes 'dust' which does not saltate within the rough range 0·15 to 0·05 mm.

To check the theory against these figures let G_0^2 in (28) and (29) be given its values 1500 and 100 (§ 1 *b*) at the limits of the transition region of figure 2. In the case of wind-blown grains it is necessary to assign some value to Θ_0 in (28). Suppose this be taken as 0·02 or twice the threshold value, and equivalent to a 'moderate wind' of 17 m.p.h. as measured at 10 m height. Comparative figures are then as follows:

	grain diameter (mm) at which		
	saltation fully developed	saltation absent	geometric mean (mm)
quartz grains in water:			
observed	2·0	0·2	0·63
predicted by (29)	1·1	0·43	0·68
quartz grains in wind:			
observed (say)	0·15	0·05	0·086
predicted by (28)	0·14	0·07	0·098

(c) *The upper diameter limit for bed ripples due to the deficit in the granular bed resistance*

Shields (1936) summarizes in a general diagram the results of much laboratory work in open channels, mostly at flow strengths close to the threshold of movement. From this diagram it can be computed that for quartz grains in water 'ripples' give place to 'scales' at about 0·5 mm diameter; 'scales' give place to 'bars' at about 0·9 mm. diameter, the bars being generally understood as flat step-like features similar to those shown in figure 7 *c*. The bars persist to the experimental limit of grain size.

Gilbert (1914) whose similar work extended to higher flow strengths, made no descriptive distinction. Bed features, whether ripples, bars or dunes, were noted in his records of the character of the bed for all grades of sand up to 0·26 cm. Records are absent for his still larger grades.

Inglis (1940, app. 2) makes a pertinent distinction that while 'dunes', which he illustrates as transverse step-like bars very similar to those of figure 7 *c*, may have (smaller) ripples on their slopes, the ripples are never complicated by any still smaller feature. On this definition he states that ripples disappear when the grain size is increased beyond 0·05 cm. No diameter limit is apparent for the dunes.

If we exclude the bars or dunes as being due to another cause, to be discussed in §13, the observed limit of bed ripples or 'scales' seems to lie between 0·09 cm (Shields) and 0·05 cm (Inglis). The geometric mean is 0·067 cm.

In the case of wind-blown grains the distinctive 'ballistic' ripple pattern should be ignored as a different phenomenon (§ 4*d*). But the usual very high value of G can be brought down into the same range as obtains in water by a sufficient reduction in grain size. The strong ballistic effect is thereby greatly enfeebled.

It is reported (Bagnold 1941, chap. 8) that when an exceptionally fine quartz sand, uniform at 0·08 mm diameter and carefully dried out, was blown in a horizontal wind tunnel, the typical sinusoidal and continuously transverse ballistic rippling was present on the grain bed over the lower range of flow strengths up to

$$u_* = 30\,\text{cm/s} \quad (\Theta_0 = \rho u_*^2/\sigma D g = 0\cdot052).$$

But at $\Theta_0 = 0\cdot052$ an abrupt change took place. The little ballistic ripples turned into a quite different and far larger system of bed features closely resembling those under a water stream (plate 2). Presumably the change marked the onset of the same secondary instability due to local turbulence effects as was dealt with in § 4*c*.

Giving G_0^2 its critical value 440 (§ 4*b*) at which $\tan\alpha_0 = \tan\alpha_-$ by figure 2, assuming $\tan\alpha_- \sim 0\cdot63$, the critical grain diameter for the onset of the same kind of bed rippling in the wind and water cases is obtained as before from (28) and (29). Comparison with the above observed figures is as follows:

	critical diameter (mm) for bed ripples	
	in water	in wind
observed (mm)	0·67	0·08
predicted (mm)	0·71 (by 29)	0·078 (by (28) and putting $\Theta_0 = 0\cdot052$)

The general agreement in all four comparisons is surprising, since the predicted figures involve no data obtained from gravity-bed experiments. It seems to justify the further application of the theory to the more practical aspects of grain flow which concern the transport rate of the grain load.

PART II. THE RATE OF GRAIN TRANSPORT OVER A GRAIN BED; THE STREAM CASE

6. Relative motion between grains and fluid

(*a*) *Transport rate as a rate of work done*

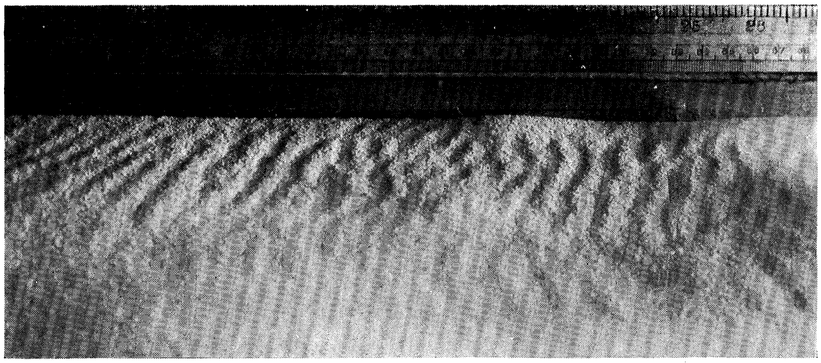
The transport rate Q as usually defined is the total mass of grains which in unit time passes unit width of a fixed cross-section of the flow. By this definition Q is directly measurable by weighing ex-fluid the quantity of grains discharged. Clearly

$$Q = \sigma \int_y^\infty CU\,\mathrm{d}y$$

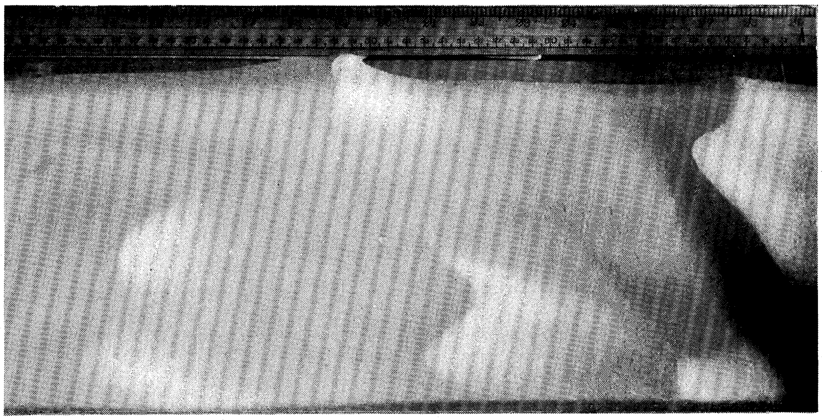
$$= \sigma D \int_h^\infty CU\,\mathrm{d}h, \qquad (30)$$

← wind direction

a



b



Effect of air viscosity on behaviour of very find wind-blown grains (diameter 80μ, quartz). Abrupt change in the bed features at the predicted critical surface stress. (*a*) Usual 'ballistic' ripples at low stresses. (*b*) Large (assumed secondary) features resembling those in water. (Reproduced from plate 10 of Bagnold 1941.)

(*Facing p.* 262)

where U is the grain velocity in the flow direction at any plane y relative to the stationary bed, and $h = y/D$.

It will be noticed that Q has the dimensions, MT^{-3}/g, of a rate of work done, in mass units, per unit width of flow. The actual rate of useful work done in transporting the grains along the bed is measurable by the tangential stress opposing their motion multiplied by their velocity relative to the bed. And the tangential stress should be proportional to the grains' normal immersed weight component. So if b is a generalized stress ratio (tangential stress opposing motion)/(normal immersed weight component), the actual rate of useful work done can be written

$$W = b\,\frac{(\sigma-\rho)\,gD\cos\beta}{\sigma D}\int_h^\infty CU\,\mathrm{d}h$$

$$= b\gamma Q/\sigma D \quad \text{by (30)}, \tag{31}$$

where γ is the unit stress $(\sigma-\rho)\,gD\cos\beta$.

The energy required to do this work may come from three sources, depending on the conditions: (i) the flow energy of the fluid; (ii) the internal, turbulent, energy of the fluid; (iii) the potential gravity energy of the grains. Of these, (i) is drawn upon only in the case of bed-load transport; (ii) only in that of a suspended load; and (iii) whenever the bed surface has a gravity slope $\tan\beta$. Since the bed-load grains dissipate energy in overcoming the tangential grain stress T, the energy drawn from source (i) has first to be transferred from fluid to grains by means of a fluid drag on the individual dispersed grains and of an associated relative motion or 'slip' in the direction of flow. This makes some discussion of the drag coefficient inevitable. For in respect of the bed load the system must be regarded as a fluid-dynamic transporting machine, and an 'efficiency' is involved.

(*b*) *The drag coefficient ψ and its dependence on grain concentration*

(*Note*. In this context the use of the conventional symbol c for this coefficient is liable to confusion with the accepted symbol C for the volume concentration. So the symbol ψ will be used instead.)

The drag coefficient ψ of a grain moving in any direction with any velocity q through a fluid is defined as the ratio $2f/\rho q^2$, where f is the stress opposing the motion per unit of the grain's aspect area.

When an isolated grain is far removed from other fluid boundaries, ψ, for spheres, is a unique experimental function of the Reynolds number Dq/ν. Various attempts have been made to define the shapes of non-spherical natural grains so that for any defined shape ψ will also be a unique function of Dq/ν. But no such definition has gained general acceptance. In what follows Heywood's (1938) experimental curve of ψ against R, for his shape coefficient 0·3, will be taken as standard for average natural grains. It is reproduced in figure 9 as the curve marked $C = 0$.

It has long been known from sedimentation experiments that the drag coefficient with regard to the local fluid velocity q past a grain increases markedly with increase in concentration. This effect has been linked quantitatively with the effect in vertical catalytic reactor columns. Here stable uniform grain concentrations can be maintained under a wide range of upward fluid velocities through them. From a recent unpublished review of this work (Zaki 1953), summarized by Richardson & Zaki (1954), I have extracted values of a modified drag coefficient ψ' which for the same local Reynolds number Dq/ν appears to define the drag on a grain at any finite concentration.

The resulting family of curves, each for a given uniform concentration, is shown in figure 9. Experimental boundary effects (column diameter) have been removed by extrapolating the given correction factor to the case of infinite diameter. The facts are already known, but have not, seemingly, been presented in this basic form. The figure indicates that as a round approximation $\psi'/\psi \sim (1-C)^{-3}$.

Figure 9 refers, of course, to grain dispersions which are undergoing random rather than laminar shear. But the increase in the drag coefficient is so large that any error introduced by this difference is likely to be small compared to the gross error involved in neglecting the effect altogether.

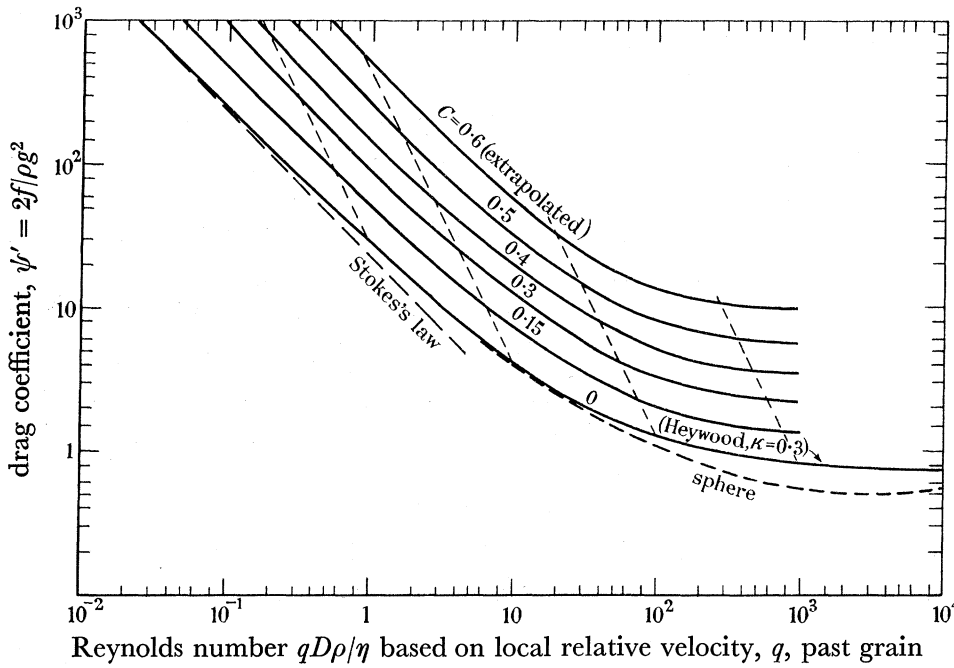


FIGURE 9. Increase in drag coefficient with grain concentration (from fluidization and sedimentation data). Relative velocity, q, found from $q = V/\epsilon$, where V is the upward fluid velocity in an empty tube at the same discharge rate, and ϵ is the porosity, $1-C$.

If in any given grain-fluid system (D and ν constant) the fluid force acting on each grain is kept constant, an increase in the concentration has the effect of lowering the local relative velocity, and consequently of lowering the value of the local Reynolds number. This effect is shown by the inclined dotted lines in the figure. As a result, at high concentrations, e.g. close to a grain bed, the effective value of R is considerably smaller than that for single isolated grains, and the gradient $d\psi'/dR$ is steeper.

(c) *Velocities relative to the fluid of the bed load and suspended grains*

Let a unit prism, situated at any distance $h = y/D$ from the bed, have unit sides, and a unit height in terms of h; and let it be assumed that the prism contains bed load and suspended grains in separate groups. (Actually each grain may possess the attributes of both kinds of load, or the two kinds may be situated within different zones.)

The numbers of grains in the bed-load and suspended-load groups are $6C_b/\pi D^2$ and $6C_s/\pi D^2$, and their total aspect areas are therefore $\frac{3}{2}C_b$ and $\frac{3}{2}C_s$. Their normal immersed weight components are $(\sigma-\rho)\,gD\cos\beta\times C_b$ and $C_s = \gamma C_b$ and γC_s respectively.

Bed-load grains

Consider the impulsion of the bed-load grains in the prism by the fluid flow. The whole of the bed-load grains exert a normal stress P_0 at the bed surface equal to their own normal immersed weight component. The associated tangential grain stress T_0 is $P_0 \tan\alpha_0$. And this results from the integrated fluid thrusts on each layer of the dispersed bed-load grains above the bed, as represented by the content of the unit prism. Equating applied and resisting tangential thrusts on the bed-load group in the prism

$$\tfrac{3}{2}C_b \times \tfrac{1}{2}\psi'_b \rho \mathscr{U}_b^2 = \gamma C_b \tan\alpha,$$

whence

$$\mathscr{U}_b = \sqrt{\left(\frac{4}{3}\frac{\gamma\tan\alpha}{\rho\psi'_b}\right)}, \tag{32}$$

where $\mathscr{U}_b$ is the relative velocity of the fluid past the bed-load grains in the direction of flow, $\tan\alpha$ is the stress ratio T/P at the plane h of the prism, and ψ'_b is the drag coefficient appropriate to the Reynolds number $\mathscr{U}_b D/\nu$ and to the local concentration C_h.

Since the bed-load grains are in normal equilibrium with the stress P they do not fall through the fluid, and their normal relative velocity $\mathscr{V}_b$ is zero.

The rate of loss of fluid-flow energy due to the presence of the bed-load grains in the prism is $\gamma C_b \tan\alpha\,\mathscr{U}_b$.

Suspended grains

These being supported by the fluid are subject to no normal grain stress P. They fall vertically through upward-moving fluid eddies with a relative velocity $\sqrt{\left(\frac{4}{3}\frac{\gamma/\cos\beta}{\rho\psi'_s}\right)}$ which has normal and tangential components

$$\mathscr{V}_s = \sqrt{\left(\frac{4}{3}\frac{\gamma\cos\beta}{\rho\psi'_s}\right)} \quad \text{and} \quad \mathscr{U}_s = \tan\beta\sqrt{\left(\frac{4}{3}\frac{\gamma\cos\beta}{\rho\psi'_s}\right)}, \tag{33}$$

where $\mathscr{U}_s$ is usually in the opposite sense to $\mathscr{U}_b$.

The loss of internal turbulent energy of the fluid in maintaining the suspension in the prism is $\gamma C_s \mathscr{V}_s$, but there is a gain of flow energy corresponding to $\gamma C_s \tan\beta\,\mathscr{U}_s$,

It will be seen that for small bed slopes the tangential bed-load relative velocity $\mathscr{U}_b$ is everywhere proportional to the terminal fall velocity $\mathscr{V}_s$ of suspended grains at the same place (same concentration and same value of ψ'/ψ), so that

$$\mathscr{U}_b/\mathscr{V}_s \sim \sqrt{\tan\alpha}.$$

(*d*) *The mean bed-load relative velocity* $\overline{\mathscr{U}}_b$; *stream case*

Figure 9 indicates that for the same Reynolds number $\mathscr{U}_b D/\nu$ and $\mathscr{V}_s D/\nu$ the drag coefficient ψ' increases some 15-fold as the proximity λ of other grain boundaries is increased from zero to the mobile limit of about 14 which corresponds to $C = 0{\cdot}53$ (figure 1). So when grains are fluid-driven over a gravity bed the increase in concentration towards the bed may be such that neither of the relative velocities $\mathscr{U}_b$ nor $\mathscr{V}_s$ are even approximately constant with distance from the bed.

But in the 'stream case' at any rate, to which the bulk of the available experimental data are confined, a constant mean relative velocity $\overline{\mathscr{U}}_b$ appears to be definable. In this case the free upward expansion of the sheared grain dispersion through the fluid is unrestricted by any upper boundary (adequate fluid flow depth); and the concentration therefore attenuates to a constant zero value at the top of the dispersion. From the argument in § 5*a*, C_0 at the base of the dispersion should also remain constant at about 0·53 under all turbulent flow conditions, except when in liquids $\Theta_0 < 0{\cdot}4$. So with this exception a mean concentration $\overline{C} = \int C\,\mathrm{d}h/C_0$ is likely to have a constant value depending only on the attenuation function.

From general observation the attenuation of concentration with height, in the absence of suspension, i.e. in the case of bed-load grains only, is far more rapid than in the case of suspended grains. A simple exponential attenuation for the bed-load grains may not be far wrong if the infinite peak is ignored. If $C \sim C_0\,\mathrm{e}^{-ah}$, the mean concentration $\overline{C}$ for $C_0 = 0{\cdot}53$ would be $0{\cdot}53\,\mathrm{e}^{-1} = 0{\cdot}2$.

When in the liquid case the number of grains moving over the bed approaches zero towards the threshold condition, $\overline{C}$ will also approach zero in the volumetric sense. But as a measure of the proximity λ of other grains, i.e. the exposed bed-surface grains, and therefore of the ratio $n = \psi'/\psi$, it is still finite. And from the discussion in § 2*a* of the saltation phenomenon in liquids it appears that the mean free distance D/λ between the moving grains and the bed surface cannot be very different from $\frac{1}{2}D$, which makes $\lambda \sim 2$. This distance also coincides with the probable limiting extent of the fluid-dynamic influence of the bed on the moving grains (Jeffreys 1929). Reference to figure 1 shows that $\lambda = 2$ is equivalent to the same volume concentration $\overline{C} = 0{\cdot}2$ as above.

From the experimental curves of figure 9 the value of $n = \psi'/\psi$ for the mean concentration $\overline{C} = 0{\cdot}2$ is found to be about 2·0.

As a working hypothesis therefore, to be tested against the experimental evidence, n will be taken as a general constant for the stream case, in whatever fluid, and to have a value of 2·0. Relation (32) now gives a mean relative velocity $\overline{\mathscr{U}}_b$ which should be a constant for any given system and definable by

$$\overline{\mathscr{U}}_b = \sqrt{\left(\frac{2}{3}\frac{\gamma\tan\alpha}{\rho\psi}\right)}, \tag{34}$$

where ψ is the conventional drag coefficient for an isolated grain uniquely appropriate to $R = \overline{\mathscr{U}}_b D/\nu$, and $\tan\alpha$ is the mean stress ratio within the bed-load dispersion.

A standard rate w_b of energy loss of the fluid, per unit of bed load, can now be defined as

$$w_b = \gamma\tan\alpha\,\overline{\mathscr{U}}_b, \tag{35}$$

which should be a constant for any given system.

7. Transport rate of fluid-driven grains

(*a*) *The characteristic bed-load transport rate*, Φ_b

In the case of the bed load the generalized stress ratio b of (31) is $\tan\alpha_0$, since all the bed-load grains ultimately rest on the bed-surface grains. So the total rate W_b of useful work done in transporting the bed load is

$$W_b = \frac{\gamma Q_b}{\sigma D}\tan\alpha_0.$$

But if the bed surface has a gravity slope $\tan\beta$, a rate of work $(\gamma Q_b/\sigma D)\tan\beta$ is contributed directly by gravity. Whence the rate W_{bF} of useful work done by the fluid is

$$W_{bF} = \frac{\gamma Q_b}{\sigma D}(\tan\alpha_0 - \tan\beta). \tag{36}$$

From § 3*a* (iii), the tangential stress applied by the fluid at the bed surface is $\mathscr{T}_F$, the total applied stress $\mathscr{T}_0$ being $\mathscr{T}_F + \gamma\chi_b\tan\beta$, where $\gamma\chi_b$ is the normal immersed weight component of the bed load (§ 3*a* (iv)). And from § 4*c* the whole non-effective applied tangential stress on the stationary bed surface (due both to form-drag and to the residual static resistance of exposed surface grains) is $\mathscr{T}_t = \gamma\Theta_t$. So the applied tangential stress due to the fluid flow, which is effective in moving the bed-load grains, is $\mathscr{T}_F - \mathscr{T}_t$.

Regarding the system as a transporting machine, the available fluid energy rate should be $(\mathscr{T}_F - \mathscr{T}_t)\times A(\mathscr{T}_F/\rho)^{\frac{1}{2}}$, where $A(\mathscr{T}_F/\rho)^{\frac{1}{2}}$ is the fluid's unimpeded velocity at the effective plane of application. Whence

$$W_{bF} = \frac{\gamma Q_b}{\sigma D}(\tan\alpha_0 - \tan\beta) = (\mathscr{T}_F - \mathscr{T}_t)\,A(\mathscr{T}_F/\rho)^{\frac{1}{2}}\,\epsilon, \tag{37}$$

where ϵ is an efficiency factor <1, which should be in the nature of an energy-rate ratio of some kind. The appropriate ratio appears to be that of the unit w_b of (35) to an ultimate unit of the form $\gamma(\gamma/\rho)^{\frac{1}{2}}$, so that

$$\epsilon = \frac{\gamma\tan\alpha\overline{\mathscr{U}}_b}{\gamma(\gamma/\rho)^{\frac{1}{2}}} = \tan\alpha\sqrt{(2\tan\alpha/3\psi)}. \tag{38}$$

Transposing w_b from ϵ in (37), we have the dimensionless relationship

$$\frac{\gamma Q_b(\tan\alpha_0 - \tan\beta)/\sigma D}{\gamma\tan\overline{\mathscr{U}}_b} = \frac{(\mathscr{T}_F - \mathscr{T}_t)\,A(\mathscr{T}_F/\rho)^{\frac{1}{2}}}{\gamma(\gamma/\rho)^{\frac{1}{2}}}$$
$$= (\Theta_F - \Theta_t)\,A\Theta_F^{\frac{1}{2}} \tag{39}$$

between the bed-load transport rate and the stress applied by the fluid at the bed surface.

The dimensionless transport-rate function on the left will be denoted by Φ_b. Evaluating $\overline{\mathscr{U}}_b$ by (34)

$$\Phi_b = \frac{Q_b}{\sigma D\sqrt{(\gamma/\rho)}}\,\frac{\tan\alpha_0 - \tan\beta}{\tan\alpha}\,\frac{1}{\sqrt{(2\tan\alpha/3\psi)}}, \tag{40}$$

which it is convenient to write as $$\Phi_b = \Phi_b'\frac{1}{B}. \tag{40a}$$

The first group $\Phi_b' = Q_b/\sigma D\sqrt{(\gamma/\rho)}$ contains all the dimensional quantities. This group has been deduced previously by a very different approach based on probabilities (H. A. Einstein 1950).

The second group $\dfrac{1}{B} = 1\Big/\dfrac{\sqrt{(2\tan\alpha/3\psi)}}{(\tan\alpha_0 - \tan\beta)/\tan\alpha}$ contains all the non-dimensional parameters not previously taken into account. $\sqrt{(2\tan\alpha/3\psi)}$ is the dimensionless part of $\overline{\mathscr{U}}_b$.

The evaluation of B requires a knowledge of the stress ratios. In liquids it seems unlikely that $\tan\alpha_0$ at the base of the dispersion can be very different from $\tan\alpha$ within the dispersion. And $\tan\alpha$ can be obtained from figure 2 for the appropriate value of G^2 as given by (28) or (29) as the case may be.

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$\overline{\mathscr{U}}_b$ and ψ can then be found in the usual way by trial and error from the experimental $R-\psi$ curve, e.g. figure 9 ($C = 0$).

In the case of wind-blown sand conditions at the bed surface appear to be somewhat different. G^2 within the dispersion is everywhere so large that $\tan\alpha$ may be taken as constant at 0·32. But bed grains are knocked along the bed surface as 'surface creep' (Bagnold 1941) by the violent bombardment of the saltation above. So at the boundary of grain movement conditions are likely to be akin to those of mechanical shearing of a mass of grains at rest; and $\tan\alpha_0$ should therefore approximate to the static value which has been taken as 0·63 from the angle of repose of average natural grains, i.e. as approximately twice $\tan\alpha$. Whence

$$\left.\begin{aligned} B_b\ (\text{liquids}) &= \frac{\sqrt{(2\tan\alpha/3\psi)}}{1-\tan\beta/\tan\alpha} \rightarrow \sqrt{(2\tan\alpha/3\psi)} \\ B_b\ (\text{wind-blown}) &= \frac{\sqrt{(2\tan\alpha/3\psi)}}{2-\tan\beta/\tan\alpha} \rightarrow \tfrac{1}{2}\sqrt{(2\tan\alpha/3\psi)} \end{aligned}\right\} \text{ when } \tan\beta \text{ can be neglected.} \quad (41)$$

The justification for the halving of B for wind-blown grains will be further discussed in § 9*e*.

We thus arrive at the important bed-load transport rate relation

$$\Phi_b = A(\Theta_F-\Theta_t)\,\Theta_F^{\frac{1}{2}}, \quad (a)$$

or

$$\Phi_b' = AB(\Theta_F-\Theta_t)\,\Theta_F^{\frac{1}{2}}, \quad (b)$$

or in terms of dimensional quantities

$$Q_b = AB\sigma D\sqrt{\left\{\frac{(\sigma-\rho)\,gD\cos\beta}{\rho}\right\}}\;(\Theta_F-\Theta_t)\,\Theta_F^{\frac{1}{2}} \quad (c)$$

$$= \frac{AB\sigma}{(\sigma-\rho)\,g\cos\beta\sqrt{\rho}}(\mathscr{T}_F-\mathscr{T}_t)\,\mathscr{T}_F^{\frac{1}{2}}. \quad (d) \qquad (42)$$

It will be noticed that

(i) The grain diameter D enters only indirectly, as a variation in the value of the drag coefficient ψ.

(ii) When the bed slope $\tan\beta$ can be neglected, as in all the experimental data to which (42) will be applied in the following section, $\mathscr{T}_F$ and Θ_F become interchangeable with $\mathscr{T}_0$ and Θ_0 of part I, and B in (41) takes its simplified form. The effect of the bed slope on the transport rate does, however, become important, e.g. locally in dune formation (§ 13) and also in transport through pipes.

(iii) In the case of wind-blown grains Θ_t is zero, and (42) becomes

$$\Phi_b = A\Theta_F^{\frac{3}{2}}, \text{ etc.} \quad (42e)$$

(*b*) *Value of the numerical constant A*

This is the only quantity not yet determined. The factor $A(\mathscr{T}_F/\rho)^{\frac{1}{2}}$ in (37) is conceived as the flow velocity u' which the fluid would have at the effective plane h' of its thrust on the moving bed load grains were these grains to be removed, i.e. were the bed surface to be made immovable.

In § 6*d* the constant drag coefficient ratio $n = 2$ was inferred for liquids at low flow strengths by assuming a mean separation distance $\frac{1}{2}D$ between grains and bed. Consistently with this, therefore, the effective thrust plane should lie at the centre of the grains concerned, or at a distance $h' = 1$. Assuming the fixed granular boundary to be 'rough' the velocity u of a turbulent fluid at a plane distant y from such a boundary is given by Keulegan (1938) from the work of Kármán, Prandtl and others as

$$u = \frac{2\cdot 3}{\kappa} u_* \log_{10}(30y/D),$$

which in this case becomes, putting the Kármán constant $\kappa = 0\cdot 4$,

$$u' = 5\cdot 75\sqrt{(\mathcal{T}_F/\rho)}\log_{10} 30,$$

whence $$A = u'/\sqrt{(\mathcal{T}_F/\rho)} = 8\cdot 5.$$

This value of A agrees well with that given by the experimental data when applied to (42), as the next section will show. But the above derivation is evidently over-simplified; because the agreement covers not only liquid-driven grains but also wind-blown grains whose effective fluid thrust height must be far greater than one diameter above the stationary bed surface. It seems that A may be a general constant for the conditions of the 'stream case'; which should indeed be true if the expression for Φ_b included all the relevant factors.

(c) *Mean bed-load grain velocity* $\bar{U}$

Let $\bar{U}$ be defined as $\int CU\,\mathrm{d}h \Big/ \int C\,\mathrm{d}h$, so that if all the bed-load grains travelled at this velocity relative to the bed, Q_b would remain unchanged. Then the normal immersed weight component $\gamma\chi_b = \gamma\int C_b\,\mathrm{d}h$ of the bed-load grains multiplied by $\bar{U}$ can be substituted for $\gamma Q_b/\sigma D$ in (39). But by (27)

$$\chi_b \tan\alpha_0 = \Theta_0 - \Theta_t = \Theta_F + \chi_b \tan\beta - \Theta_t,$$

or $$\gamma\chi_b(\tan\alpha_0 - \tan\beta) = \mathcal{T}_F - \mathcal{T}_t. \qquad (43)$$

Dividing (39) by (43) $$\bar{U}_b = \frac{A(\mathcal{T}_F/\rho)^{\frac{1}{2}}}{(\gamma/\rho)^{\frac{1}{2}}} \tan\alpha\,\bar{\mathscr{U}}_b$$

$$= A\Theta_F^{\frac{1}{2}} \tan\alpha\,\bar{\mathscr{U}}_b. \qquad (44)$$

It may be noted that (44) gives directly the bed load χ_b in terms of Φ_b as

$$\chi_b = \frac{\Phi_b}{A\Theta_b^{\frac{1}{2}}(\tan\alpha_0 - \tan\beta)}, \qquad (45)$$

(d) *An analogous characteristic suspended-load transport rate*, Φ_s

Suspended grains may be assumed statistically to be falling through the fluid with a relative velocity whose normal component is $\bar{\mathscr{V}}_s$. So the maintenance of a steady suspended load over a horizontal bed should require useful work to be done at a rate $\gamma\chi_s\bar{\mathscr{V}}_s$, where $\gamma\chi_s$ is the normal immersed weight component. This rate appears to be identical with that required to push an equal frictionless grain load up an imaginary slope of $\bar{\mathscr{V}}_s/\bar{U}_s$. So the ratio $\mathscr{V}_s/\bar{U}_s$ can be regarded as the ratio b in (31), i.e. as the analogue of $\tan\alpha_0$ in case of the bed load.

By analogy therefore it seems possible to conceive of a characteristic suspended load function

$$\Phi_s = \frac{Q_s}{\sigma D}(\mathscr{V}_s/\bar{U}_s - \tan\beta)\Big/\frac{\mathscr{V}_s}{\bar{U}_s}\mathscr{V}_s$$

$$= \frac{Q_s}{\sigma D\sqrt{(\gamma/\rho)}}\frac{1}{B_s}, \tag{46}$$

where $B_s = \sqrt{\left(\frac{4}{3n\psi_s}\right)}\Big/\left(1-\frac{\bar{U}_s}{\mathscr{V}_s}\tan\beta\right)$, which becomes $\sqrt{(4/3\psi_s)}$ when $\bar{U}_s\tan\beta \ll \mathscr{V}_s$.

From general experience the concentration of suspended loads under stream-case conditions is very small. In rivers it rarely exceeds 1 %, and is probably far less in the atmosphere even in a dust storm. So it is likely that n can be taken as unity, the suspended grains being regarded as isolated. And so long as the $\tan\beta$ term can be neglected $\mathscr{V}_s = \sqrt{\left(\frac{\gamma}{\rho}\frac{4}{3\psi_s}\right)}$ and $B_s = \sqrt{(4/3\psi_s)}$. It will be noticed, however, that whereas B_b in (41) is a constant for the system, supposedly independent of the flow velocity, B_s should begin to increase very rapidly as $\bar{U}_s\tan\beta \rightarrow \mathscr{V}_s$.

The general conditions under which a given turbulent fluid flow becomes capable of maintaining a suspension of grains of a given size and density are still obscure. But the supply rate of the necessary internal turbulent energy to the fluid is clearly associated with the fluid shear due to the drag at its flow boundaries; and this in turn depends on the physical nature of these boundaries.

In the two-dimensional case the only flow boundary is the gravity bed. When a liquid flows over a loose grain bed this boundary, at the threshold of grain movement, is an ordinary fixed boundary of granular roughness; and the nature of the fluid turbulence is determined accordingly. But as the flow is increased the nature of the boundary changes progressively, till at a flow assumed in part I to correspond, for liquids, to $\Theta_0 \sim 0{\cdot}4$, the boundary to the fluid flow has become a zone of moving bed-load grains. So over this range of Θ the nature of the turbulence may be expected to be transitional, changing progressively with the nature of the boundary. And its capacity to maintain a suspended grain load should also be transitional.

But when the critical bed stress $\Theta_0 \sim 0{\cdot}4$ is exceeded, the liquid flow has virtually lost cognizance of the underlying stationary boundary. The shear resistance is wholly granular, and Θ_t is zero. The transition should therefore be complete. So if grain suspension occurs at all, it would be expected to have become fully developed at the onset of the new and continuing boundary conditions. Φ_s should become proportional to $\Theta^{\frac{3}{2}}$ which would be the new measure of the rate of available energy supply for the transport of both bed and suspended load, and to which Φ_b should also be proportional.

It is conceivable, further, in the two-dimensional case where the energy supply to the fluid turbulence arises wholly from the act of transporting the bed load, that the energy distribution is such that the work rate represented by Φ_s is equal to that represented by Φ_b. This concept is admittedly speculative, but it is worth pursuing a little further because it will be found to give results in surprising agreement with the experimental data. (See §8*b* under *Total transport rate*.)

Φ_s and Φ_b are expressed in different units B_s of (46) and B_b of (41), and are not additive. But equal work rates can be expressed in terms of Φ_b by

$$\Phi_{s(b)} = \Phi_b(B_s/B_b), \tag{47}$$

and the whole transport rate Φ resulting from putting experimental values of the whole measurable quantity $Q = Q_s + Q_b$ into (42*a*) should be given by

$$\Phi = \Phi_b(1 + B_s/B_b) = A\Theta_F^{\frac{3}{2}}(1 + B_s/B_b). \tag{48}$$

In the case of a three-dimensional channel, while the energy supply for the bed-load transport can arise only from the shearing due to the bed resistance, further energy may be supplied to the internal fluid turbulence on account of the resistance of other boundaries. It is not clear how far this is likely to affect the capacity of the flow to support suspended grains.

It will be noticed that the occurrence of $\overline{U}_s \tan\beta$ in the expression (46) for B_s allows of an unlimited increase in the transport of a suspended load in relation to the bed load in the case of fine particles or of large flow velocities. The denominator $1-(U_s/\mathscr{V}_s)\tan\beta$ appears as a modulus representing the flow conditions. Q_s should be proportional to Q_b only if the flow is varied in such a way that $\overline{U}_s \tan\beta$ is either negligible or constant.

If the above concept is sound the actual transport rate Q_s of a fully developed suspended load should be given by

$$Q_s = \Phi_s \sigma D \sqrt{(\gamma/\rho)}\, B_s = Q_b(B_s/B_b). \tag{49}$$

The suspended load $\gamma\chi_s$ would be given by $\gamma Q_s/\sigma D\overline{U}_s$. But no expression is obtainable for $\overline{U}_s$, though it might be estimated approximately in some cases from the fluid-flow velocity. If the bed has a gravity slope there must be a tangential stress contribution $\gamma\chi_s \tan\beta$ due to the suspended load, which is applied to the fluid above the bed-load zone. So it seems that the fluid stress Θ_F operating on the bed load will thereby be augmented, with a consequent increase in Φ_b. This increase on account of the suspended load would usually be insignificant. But it may well affect the value of $\partial Q/\partial \tan\beta$ along the flow, with which dune formation appears to be concerned (see §13).

8. Comparison of theoretical transport rates with experimental data

(*a*) *Wind-blown sands; bed load only*

There is no evidence that wind-blown sand grains, of sizes found in dunes and as opposed to the smaller dust particles, are ever appreciably suspended by atmospheric turbulence. Indeed, they are so relatively massive, and travel so far through the fluid by their own inertia, that suspension seems very unlikely. The quantitative transport-rate data will therefore be assumed to refer to bed load only.

I found (Bagnold 1936) that the transport rate Q_b varied as $\mathscr{T}^{\frac{3}{2}}$ upwards from the threshold of grain movement, in conformity with (42*e*). The empirical relation originally found was

$$Q_b = A'\mathscr{T}^{\frac{3}{2}} g\rho^{\frac{1}{2}},$$

where $A' = 1{\cdot}5(D/0{\cdot}025\,\text{cm})^{0{\cdot}5}$ and is therefore a variable parameter. Dividing both sides by $\gamma^{\frac{3}{2}}$ this becomes

$$\Phi_b' = A'\Theta^{\frac{3}{2}}, \tag{50}$$

since the buoyancy factor $\gamma/\sigma D$ is in this case unity.

Note. In what follows the effect of the gravity slope $\tan\beta$, if any, can be neglected, so $\Theta_F = \Theta_0$; and since the applied tangential stress is in all cases that at the bed surface, all suffixes to Θ will be dropped.

In these experiments all the sand moved was collected and weighed. So the values of Φ_b' should not be in doubt. But $\mathscr{T}$ was inferred only, from measurements of the wind-velocity gradient, assuming the Kármán constant κ had the value 0·4 for grainless fluids. In view of later work, e.g. Vanoni (1946) and H. M. Ismael (1951), κ may in reality have been considerably smaller owing to the presence of dispersed grains within the zone of measurement. The numerical coefficient 1·5 of A' should therefore be smaller. If so some unknown diameter effect on κ may have introduced a systematic error leading to a false value of the index 0·5 of A'.

Zingg (1950) repeated my experiments over a range of grain sizes from 0·02 to 0·07 cm, and found A' to be given empirically by a relation corresponding to

$$A' = 0{\cdot}72(D/0{\cdot}025\,\text{cm})^{0{\cdot}75}.$$

Here the doubt about the magnitude of the applied stress was removed by direct measurement. But the transport rate was inferred only, by the extrapolation down to the bed of the weights of sand collected at a range of heights above it. So the surface creep of grains along the bed which I had found to amount to some 25 % of the whole may possibly have been missed. This may also have caused a systematic error leading to a false value of the index.

The only fair compromise between the two discrepant results is to take the geometric means of both coefficients and indices and to write

$$A' = 0{\cdot}93\left(\frac{D}{0{\cdot}025\,\text{cm}}\right)^{0{\cdot}62},$$

as shown by (*c*) in figure 10.

Now if the transport rate is expressed as the group of dimensional quantities $\Phi_b' = \dfrac{Q_b}{\sigma D\surd(\gamma/\rho)}$ in (40) there should be no variation of Φ_b' with grain diameter, since D occurs as $D^{\frac{3}{2}}$ in the denominators on each side of (39); and no explanation of the large variation found experimentally has previously been forthcoming.

But the introduction of the parameter B makes such a variation possible. And if the foregoing theory is correct the experimental parameter A' should be proportional to B as given by (41).

The values of B, calculated from Heywood's ψ-R relation given in figure 9 ($C = 0$) and from the assumption that $\tan\alpha$ for wind-blown sands is constant at 0·32 (§1*b*), are plotted in figure 11 against grain diameter. They are repeated in column 3 of table 1 for the experimental diameter range. The compromise experimental values of A' from figure 10 are shown in column 4.

The ratio A'/B which gives the experimental value of the numerical constant A in (42) is shown in column 5. As will be seen it is nearly constant and agrees well with the theoretical estimate of 8·5 made in §7*b*.

Multiplying the mean value 8·75 of A from column 5 by B from column 3, the resulting values of A' are plotted back on to figure 10 as the curve (*d*). The discrepancy between

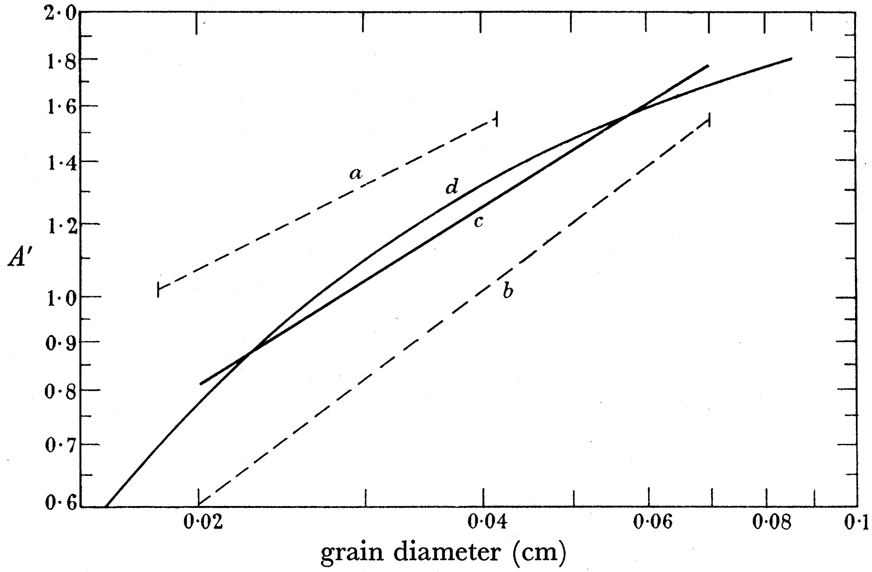

FIGURE 10. Wind-blown sand: experimental and calculated variation of transport rate with grain size. (*a*) Bagnold (1936); (*b*) Zingg (1950); (*c*) compromise experimental relation; (*d*) calculated relation. Experimental relation, $\Phi'_b = Q_b/\sigma D\sqrt{\gamma/\rho} = A'\Theta_b^{\frac{3}{2}}$. Calculated relation, $A' = AB$, where A is a constant and B, in figure 11, is determined by the mean relative velocity $\overline{U}_b$ in which the mean drag coefficient $\overline{\psi}$ is taken as twice that for an isolated grain. Experimental values of A' give the constant A a value of 8·75.

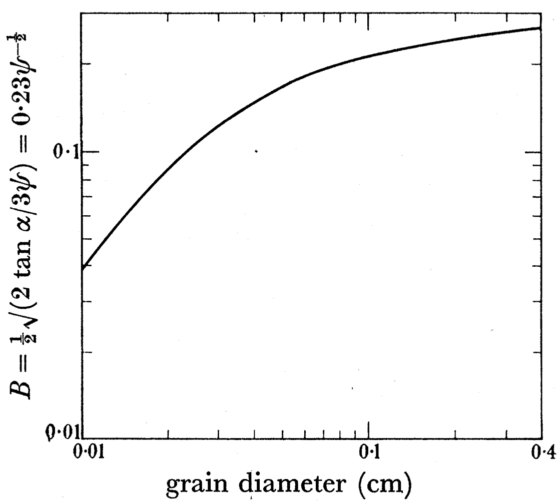

FIGURE 11. Values of $B = \frac{1}{2}\sqrt{(2 \tan \alpha/3\psi)}$ for wind-blown grains, where ψ is the drag coefficient for a single isolated grain and $\tan \alpha = \text{constant} = 0{\cdot}32$ for elastic grains, e.g. quartz.

TABLE 1

1 grain diam. (cm)	2 drag coeff. ψ (on $\overline{\mathscr{U}}_b$)	3 $B = \frac{1}{2}\sqrt{(2 \tan \alpha/3\psi)}$	4 A' (experimental)	5 $A = A'/B$
0·02	6·22	0·087	0·81	9·1
0·03	3·45	0·1245	1·04	8·26
0·04	2·45	0·148	1·24	8·4
0·05	1·92	0·167	1·42	8·5
0·06	1·63	0·181	1·6	8·83
0·07	1·44	0·192	1·76	9·15

the compromise emprical power relation of A' to D and the calculated relation given by the value of B nowhere exceeds 5 % within the experimental diameter range.

Thus the anomaly of a variation of transport rate with grain size is accounted for quantitatively by the variation in the drag coefficient ψ provided, as has been assumed from the reasoning of § 6 *d*, the effective value of ψ under stream case conditions is taken as twice the value for an isolated grain.

If the theory is sound, and A is a general constant for the stream case, it should now be possible to predict the transport rate of water-driven grains also. The only additional information needed for this is the value of the experimental parameter Θ_t (§ 4 *c*) which denotes the ineffective bed resistance peculiar to liquid-driven grains at low applied stresses. This is obtainable from the experimental results themselves.

(*b*) *Water-driven grains; bed load plus suspended load*

The only data readily applicable to the present two-dimensional case comes from laboratory experiments in open rectangular channels. In other cases, e.g. grain transport through pipes, it is not yet possible to separate the drag $\mathscr{T}$ at the surface of the grain bed from that due to other, fixed, parts of the flow boundary. In open channels where this additional drag is reasonably small it can be estimated approximately and a suitable correction made. In what follows all the experimental figures of the measured overall boundary drag, obtained from records of gravity slope and water depth, have been reduced by a common factor $b/(b+2d)$, where b and d are the bed breadth and wetted wall height. In reality the reducing factor is likely to be a variable depending on relative grain size and wall roughness and also on the magnitude of the form-drag if any. But any attempt at refinement involves arbitrary and uncheckable assumptions. Most of the rather large scatter in the experimental plots which follow is no doubt to be attributed to this cause, since the data cover wide ranges of breadth/depth ratio.

Laboratory studies of the relation between the estimated bed drag $\mathscr{T}$ and the overall transport rate Q have been discontinued in recent years. The data to be used here comes from Gilbert (1914) and U.S. Waterways (1935). The former refers to sands of nearly uniform size, and the latter to naturally graded sands.

In figures 13, 14 and 15 the experimental figures for the over-all transport rate Q, which may include suspended load, are plotted in terms of $\Phi' = Q/\sigma D\sqrt{(\gamma/\rho)}$ against the corrected tangential bed stress $\mathscr{T}$ in terms of $\Theta = \mathscr{T}/\gamma$. The data from the two sources have been amalgamated as far as possible, that for sands having the same value of B_b as evaluated below being plotted together. It will be seen that the fit is generally good, though in plots V and VI the U.S.W. data appears somewhat anomalous, perhaps owing to a systematic error in the flow depth factor in $\mathscr{T}$ arising from the difficulty of estimating the mean plane of a markedly rippled bed surface.† In the case of Gilbert's data which alone extends to $\Theta > 0{\cdot}4$ the Froude number of the flow extended well beyond unity into the supercritical condition. Distinguishing symbols have been used, from the ample overlap of which it appears that the coincidence noticeable at first sight between the critical Froude number and the sudden change in the trend of the plots is due merely to the accidental dimensions of the apparatus used.

† Or to the plane floor being allowed in places to become bare of an overlying sand bed, see § 12 *a*.

The very clear-cut discontinuities at the bases of all the plots except II for the largest grains has already been dealt with in § 4*c* and attributed to the onset of secondary bed instability resulting in secondary form-drag.

If no suspended load is being transported $\Phi' = \Phi'_b$ should be given by (42) as

$$\Phi' = A'(\Theta - \Theta_t)\,\Theta^{\frac{1}{2}}, \tag{51}$$

where $A' = 9B_b$ taking A at a round number, and the experimental parameter Θ_t has perforce to be taken from the plots themselves as the value of Θ for a vanishing value of Φ' (either branch).

If suspension occurs, it should be fully developed when $\Theta > 0{\cdot}4$. Θ_t should by then have vanished owing to the high grain concentration at the base of the bed load. So the total value of Φ' should be given by (48) as

$$\Phi' = A'\Theta^{\frac{3}{2}}(1 + B_s/B_b). \tag{52}$$

Values of B_b and B_s

Tan α in (41) for B_b is now variable with grain diameter, because G^2 as given by (29) ranges over the transition region of figure 2. The values of G^2 from (29) and of tan α from figure 2 are shown in columns 2 and 3 of table 2 for a range of sizes of quartz grains in water.

TABLE 2

1 diameter (cm)	2 G^2	3 tan α	4 ψ_b	5 B_b	6 $A' = 9B_b$	7 $\sqrt{(4/3\psi_s)}$
0·01	1·2	0·75	90	0·075	0·68	0·2
0·02	9·8	0·75	16	0·177	1·59	0·46
0·03	33	0·75	6·4	0·28	2·52	0·62
0·04	78	0·75	4·0	0·353	3·18	0·75
0·05	153	0·74	2·9	0·412	3·71	0·83
0·07	420	p·62	2·15	0·439	3·95	0·95
0·1	1220	0·47	1·7	0·428	3·85	1·09
0·15	4100	0·32	1·35	0·400	3·6	1·175
0·2	—	0·32	1·17	0·427	3·84	1·22
0·4	—	0·32	0·9	0·486	4·37	1·3?

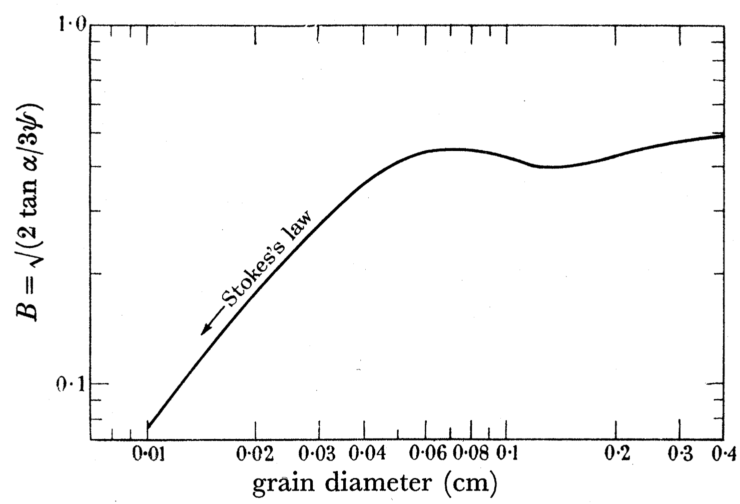

FIGURE 12. Values of B for quartz grains in water. The curve differs from that of figure 11 for wind-blown grains because tan α is now a variable and increases through the size range from 0·32 for large grains to 0·75 for small grains.

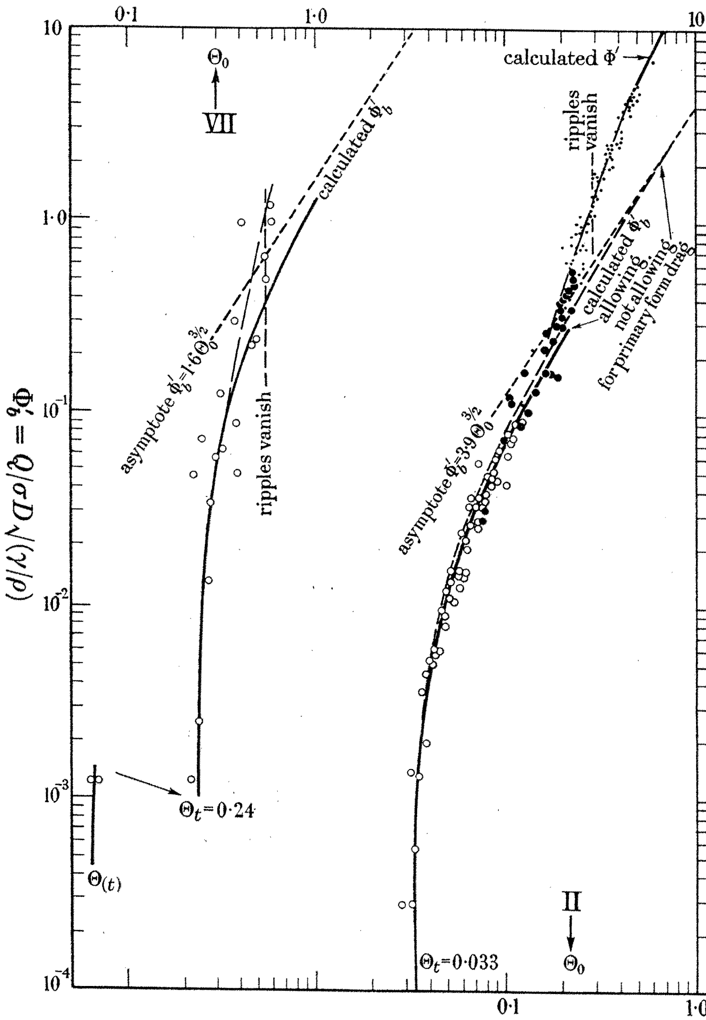

FIGURE 13. Transport rates. Plot II: ○, U.S.W., 0·0586 cm; ●, Gilbert, 0·0787 cm, subcritical flow; ·, Gilbert supercritical flow. Plot VII: U.S.W., 0·0205 cm. Showing close agreement at low values of Θ_0 between experimental rates of total load with calculated rates for bed rate alone as given by $\Phi_b' = 9aB(\Theta_0 - \Theta_t)\,\Theta_0^{1/2}$, where the constant 9 comes from wind-blown sand data; $a = 1$ except for plot II (primary form-drag only, $a = 0{\cdot}9$); B is given by figure 12, calculated from a mean drag coefficient $\psi' = 2\psi$; and Θ_t is the experimental parameter given by the value of Θ_0 at the foot of the plot.

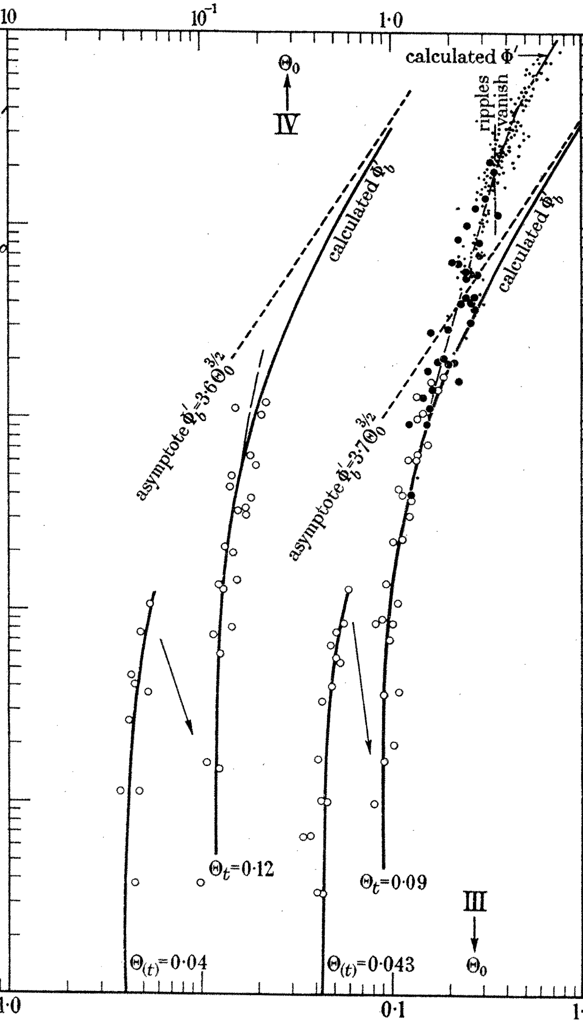

FIGURE 14. Transport rates; plots III and IV.

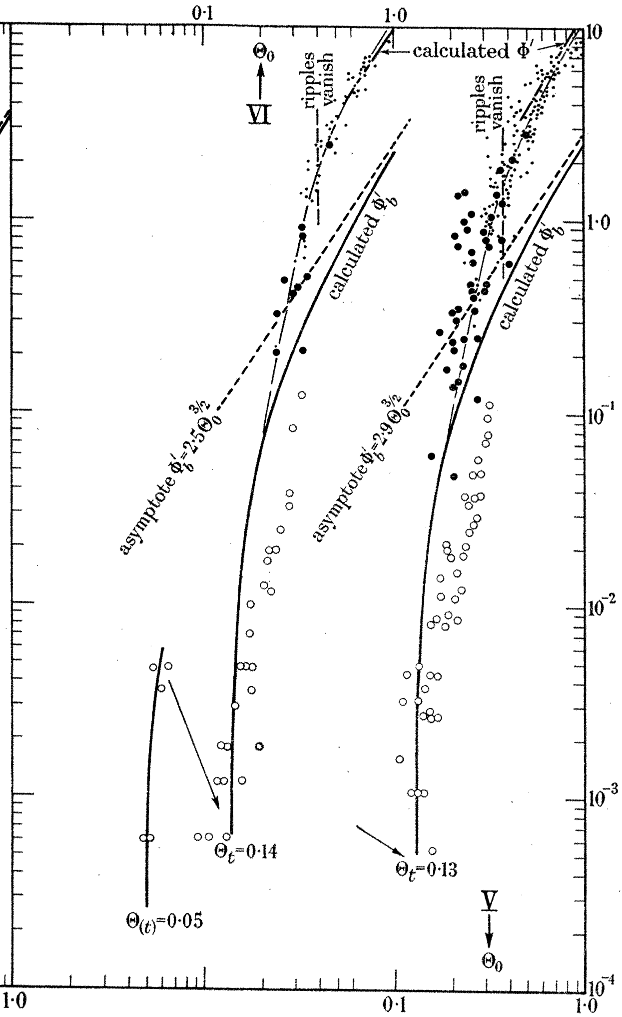

FIGURE 15. Transport rates. Plot V: ○, U.S.W., 0·347 cm; ●, Gilbert, 0·0376 cm, subcritical flow; ·, Gilbert, supercritical flow. Plot VI: ○, U.S.W., 0·0310 cm; ●, Gilbert, 0·0305 cm, subcritical flow; ·, Gilbert, supercritical flow. The U.S.W. data seem anomalous in these two cases, possibly owing to an error in the difficult estimation of the mean water depth over a deeply rippled bed.

The elements ψ_b and B_b in columns 4 and 5 result from evaluating $\overline{\mathscr{U}}_b$ by (34) as before. $A' = 9B_b$ appears in column 6. The numerator $\sqrt{(4/3\psi_s)}$ of B_s in (46) appears in column 7. The denominator, which may or may not be less than unity, has yet to be estimated.

The curve of B_b against grain diameter is shown in figure 12. Comparing this with figure 11 for wind-blown grains, the effect of the variation of $\tan\alpha$ through the transition region is seen to make the curve flatten out as D exceeds about 0·05 cm. But no great accuracy is possible in the measurement of $\tan\alpha$ from figure 2.

Bed-load transport

The lower continuous curves superimposed on plots III to VII are those calculated from (51) by giving B_b the appropriate values from figure 12. Though the footings of these curves are deliberately positioned by the values of Θ_t it will be seen that their upward courses coincide remarkably well with the plots for considerable ranges.

In plot II the curve given by (51) is shown broken. This plot does not show the discontinuity of the others. The grain bed was, however, reported as rippled. And since the grain size of 0·0586 cm is close below the critical diameter at which rippling should cease (§ 5 *c*) the form-drag is likely to be that due to primary rippling only. The effective stress $(\Theta - \Theta_t)$ should therefore be reduced by the correcting factor a in (26). Taking $G^2 \sim 300$ from table 2 for this diameter, figure 2 gives $\tan\alpha$ as about 0·71. For this the factor a in (26) is found to be 0·9. The continuous curve results from this reduction. The agreement with the experimental values is now very good.

This general agreement suggests strongly that at low stresses the fluid turbulence is incapable of causing any appreciable suspension, even of the fine 0·02 cm grains. It also suggests that the sudden increase in the experimental values of Φ' from those given by (51) must be due to the development of a suspended load, additional to the bed load. If so, the bed-load transport rate Φ'_b should continue to be given by (51) at higher values of Θ. But if Θ_t disappears at about $\Theta = 0·4$ the curves of (51) probably run into their asymptotes $A'\Theta^{\frac{3}{2}}$ at about this stress.

Total transport rate

The evaluation of the total rate from (52) requires a knowledge of the denominator $(1 - \overline{U}_s \tan\beta/\overline{\mathscr{V}}_s)$ of B_s. In a shallow laboratory channel $\overline{U}_s$ should be comparable in magnitude to the mean water velocity $\bar{u} =$ (discharge/cross-section area). The value thus given will be somewhat too big owing to increasing grain concentration in the lower layers. But the same increase in concentration will make the value of $\overline{\mathscr{V}}_s$ also too big if this is taken as the fall velocity $\mathscr{V}_s = \sqrt{(\gamma/\rho)}\sqrt{(4/3\psi_s)}$ of an isolated grain. So the ratio $\bar{u}/\mathscr{V}_s$ may be a fair approximation to $\overline{U}_s/\overline{\mathscr{V}}_s$.

Values of $\bar{u}\tan\beta$ have been taken from Gilbert's data for each of his readings where $\Theta > 0·4$. These have been averaged separately for each group of readings covering the ranges 0·4 to 0·5, 0·5 to 0·6, etc., of Θ, and divided by $\mathscr{V}_s$. The resulting figures for the denominator $(1 - \overline{U}_s \tan\beta/\mathscr{V}_s)$ are shown in table 3 *a*. From this the factor

$$(1 + B_s/B_b) = 1 + \sqrt{(4/3\psi_s)}/B_b(1 - \overline{U}_s \tan\beta/\overline{\mathscr{V}}_s)$$

results, and is shown in table 3 *b*.

Relation (52) can apply only when $\Theta > 0{\cdot}4$. But when its factor $1 + B_s/B_b$ is given the above values, the pieces of continuous curve result which have been superimposed at the top of each of the relevant plots II, III, V and VI. The very close agreement in all four cases is striking; the more so since in this case the locations of the calculated curves are entirely inferential.

It seems likely that the approximation $\overline{U}_s/\mathscr{V}_s \sim \bar{u}/\mathscr{V}_s$ would only hold if the ceiling of the suspended grains lay close to the free water surface. Some other approximation for $\overline{U}_s$ would be necessary for greater relative water depths.

TABLE 3

(*a*) Values of $1 - \overline{U}_s \tan\beta/\mathscr{V}_s$ (applicable to mid-range of Θ)

range of Θ …	0·4–0·5	0·5–0·6	0·6–0·7	0·7–0·8	0·8–0·9
diam. (cm)					
0·0787	0·82	0·80	0·77	—	—
0·0507	0·845	0·815	0·833	0·785	0·722
0·0376	0·823	0·823	0·76	0·81	0·656
0·0305	0·875	0·84	0·77	0·807	0·77

(*b*) Values of the factor $1 + B_s/B_b$ in (52)

0·0787	3·6	3·8	3·9	—	—
0·0507	3·46	3·55	3·5	3·65	3·88
0·0376	3·85	3·85	4·1	3·9	4·58
0·0305	3·54	3·64	3·9	3·8	3·9

Gilbert's paper of 40 years ago appears still to provide the only relevant data. Short of new and specially designed experiments it might be profitable to examine in detail the correlation between the scatter of his individual Φ' values in the region $\Theta > 0{\cdot}4$ and the corresponding scatter of $\bar{u}\tan\beta$. The average values of $\bar{u}\tan\beta$ in table 3*a* cover wide variations within the same narrow range of Θ on account of Gilbert's repetition of his experiments with greatly differing combinations of bed slope and water depth.

Comparing the plots for the different grain sizes, it seems significant that suspension by fluid turbulence appears in every case to begin suddenly at $\Phi' = 0{\cdot}1$. This rather confirms the view that the nature of the turbulence needs to be modified by some definite degree of bed-load development before suspension becomes possible. It may also be significant that in every case the suspension begins at a value of Θ about 4 times the plane-bed threshold $\Theta_{(t)}$. If, as White (1940) suggests, $\Theta_{(t)}$ itself may diminish with increase of flow depth and consequent increase in the scale of the general turbulence, it is likely that the onset of suspension may follow suit.

9. FURTHER DISCUSSION OF THE TRANSPORT-RATE EVIDENCE

(*a*) *General transport-rate picture for the stream case*

Experimental plots such as II to VII of figures 13 to 15 are not mutually comparable, because the scales B_b of the purely dimensional group Φ' vary with grain size. But by converting Φ' into Φ by dividing each set of Φ' values by the appropriate value of B_b, all the rates of grain transport under stream case conditions, in whatever fluid, should be comparable as regards Φ_b for the bed loads. And all the asymptotes $\Phi_b = 9\Theta^{\frac{3}{2}}$ should coincide.

Figure 16 results. To the smoothed plots II to VII for water-driven quartz grains the data for the whole experimental range of wind-blown grains has been added as curve I. As has been shown this is now a single-value relation within the limits of experimental error. The data from Bagnold (1955) has also been added as curve VIII. This refers to experiments at the opposite end of the density scale. The relative density ratio $(\sigma-\rho)/\rho$ was made as low as 0·004, which compares with 2400 for the wind-blown grains.

Finally, since all the curves appear to be asymptotic to $\Theta^{\frac{3}{2}}$, the picture can be further simplified by re-plotting against a new stress function $\Theta_* = \{(\Theta-\Theta_t)\,\Theta^{\frac{1}{2}}\}^{\frac{2}{3}}$ as shown in figure 17. The development of the suspension can now be seen unmasked by the other curvature due to the subtractive constant Θ_t. From this it appears that according to the experimental evidence the general transport rate of fluid-driven grains under stream case conditions may be approximately expressible by the single relation

$$\Phi = 9\Theta_*^{\frac{3}{2}}\left(1+\frac{B_s}{B_b}\right). \tag{53}$$

Here B_s is to be taken as zero from the threshold up to about $\Theta = 4\Theta_{(t)}$ for liquid-driven grains and up to the experimental limit of $\Theta \sim 0{\cdot}2$ for wind-blown grains. And for liquid-driven grains at any rate, B_s becomes definable by (46) from $\Theta = 0{\cdot}4$ upwards.

(*b*) *Gaps in the experimental evidence*

The evidence covers a very limited range of grain size and often a too limited range of Θ. According to § 7*d* suspension, if it occurs at all, ought to be fully developed at the critical value of Θ at which the stationary-boundary drag Θ_t vanishes, and at which consequently the nature of the fluid turbulence should be fully modified to the conditions of a moving-grain boundary. The evidence of figures 13 to 17 confirms this surprisingly well for the predicted critical value $\Theta = C_-\tan\alpha_- \sim 0{\cdot}4$ for liquid-driven grains. But in the case of wind-blown grains ($D > 0{\cdot}01$ cm at any rate) for which the number G is large the grain bed is so disturbed by the ballistic bombardment of the saltation that Θ_t vanishes at the threshold of grain movement. This is confirmed by the onset of the $\frac{3}{2}$ power relation at the threshold (see figure 16). Suspension ought therefore to be fully developed at the threshold. But it is not. And though, as can be seen from figure 16, the experimental range of Θ does not extend quite far enough to show whether suspension does develop later conformably with water-driven grains, there is little reason why it should. It looks therefore as if, as the effects of grain inertia are increased with increase of grain size in liquids, a condition would be reached at which suspension ceases to take place at all. Experiments with larger grains in water, and up to stresses say $\Theta = 0{\cdot}3$, are needed to clear this up. Gilbert's data for grain sizes larger than 0·0787 cm are inconclusive.

There is also a serious lack of quantitative experimental information about the transport of fine grains. Obvious complications arise from cohesion when the diameter is too far reduced. But cohesion appears not to be appreciable when $D \nless 60\mu$. The gap to be filled by comparative experiments covering the same stress range and under the same conditions as Gilbert's is therefore between 60 and 300μ. (Control of grain size, formerly difficult in the case of fine material in adequate quantity, can now be easily achieved by modern cyclone separation such as is used in ore-dressing techniques.)

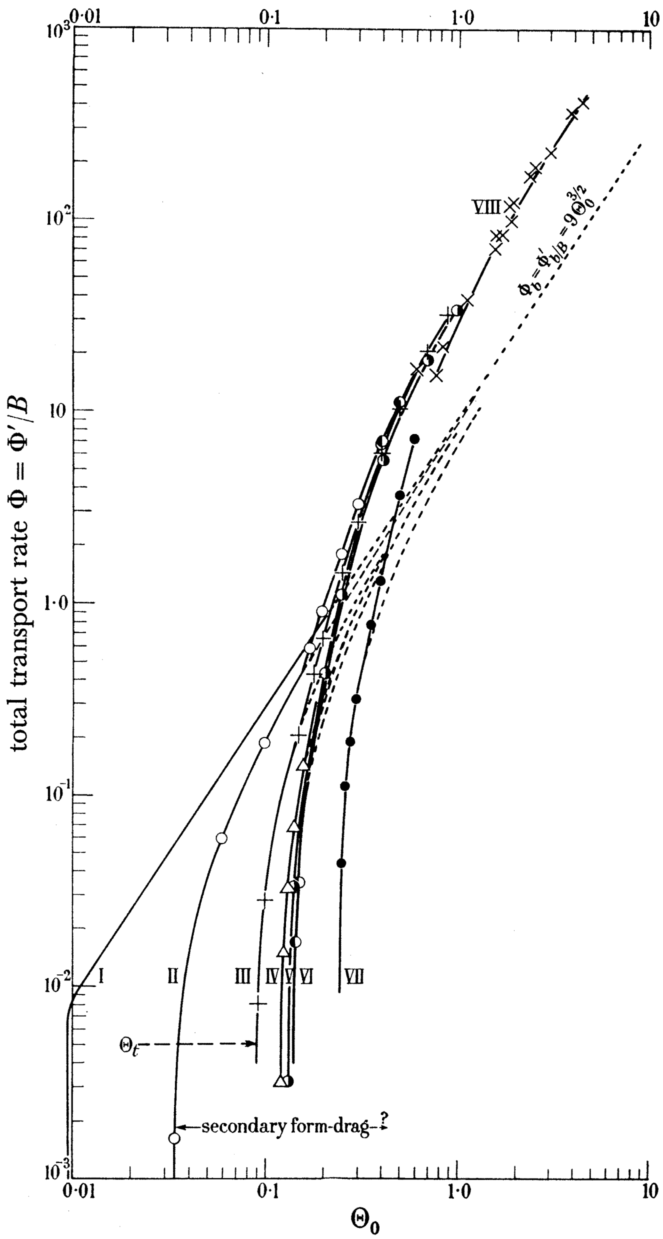

FIGURE 16

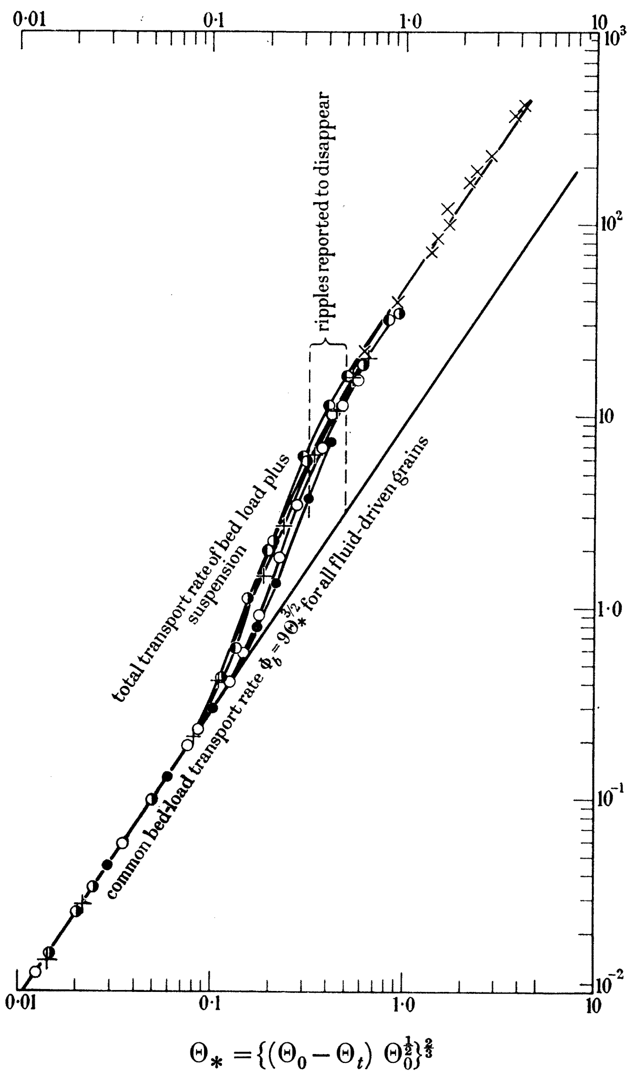

FIGURE 17

FIGURE 16. Transport rates plotted in terms of the general function $\Phi = \Phi'/B$. I. Experimental data for wind-blown sand; values of B from figure 11. II to VII. Experimental data for quartz grains in water; mean curves of figures 13 to 15 re-plotted with ordinates Φ' divided by the appropriate values of B from figure 12. VIII. Wax-lead stearate grains ($D = 0{\cdot}136$ cm) in water and Perspex grains ($D = 0{\cdot}158$ cm) in brine; density difference $\sigma - \rho = 0{\cdot}004$; from data given in Bagnold (1955). Calculated extrapolations of transport rates of bed load alone are shown by the dashed curves; all these are now asymptotic to $\Phi = 9\Theta_0^{\frac{3}{2}}$. Departure of the experimental plots from the bed-load curves are assumed to be due to the development of a suspended load in addition to the bed load. The plotting symbols connect with those of figure 17.

FIGURE 17. Re-plot of figure 16 in terms of $\Theta_* = \{(\Theta_0 - \Theta_t)\,\Theta_0^{\frac{1}{2}}\}^{\frac{2}{3}}$ showing the common range over which the suspended load appears to develop, the narrow range of Θ_* over which the bed features disappear, and the apparent ultimate proportionality of the transport rates of bed load and suspended load. Plotting symbols connect with figure 16: ○, II; +, III; ◑, V; ◐, VI; ●, VII; ×, VIII.

(c) *Similarity of grain behaviour*

From §§ 4 and 5 the general order of the number G appears to determine bed-surface effects such as saltation and rippling. But the number B_b which includes G implicitly in the value of $\tan\alpha$ seems likely to be a more discriminating criterion of grain behaviour. Compare, for instance, curve VIII of figure 16, for large but very light wax grains with curve VII for much finer quartz grains. From the details $(\sigma-\rho)/\rho = 0{\cdot}004$, $D = 0{\cdot}136$ cm, $\tan\alpha = 0{\cdot}75$, B_b for curve VIII is found to be 0·157. By figure 12 this value is that for quartz grains of diameter 0·018 cm, which is close enough to the 0·02 cm of curve VII for the trends of the two curves to be compared. It seems likely that for the same relative bed stress, in terms of Θ, grains giving the same values of B_b would behave alike, provided that within the suspension range of Θ, the term $1-(\overline{U}_s \tan\beta/\mathscr{V}_s)$ is reasonably constant.

(d) *Disappearance of bed ripples with increase in Θ; significance of the critical stress $\Theta = C_- \tan\alpha_- \sim 0{\cdot}4$*

The theory of the primary cause of bed-ripple formation under liquid flow outlined in § 4*a* required that this cause should be removed with the disappearance of the fluid resistance τ'/γ at the bed surface. From the source literature it is found that in all the relevant experiments, in which Θ was made sufficiently large, i.e., in plots II, III, V, VI and VII, the bed ripples were in fact reported to disappear, and the bed to become smooth, when Θ was increased to the values indicated on the plots (figures 13 to 15). And except for plot II where the rippling was reported as doubtful at best, all the values of Θ are close to 0·4.

It is true that Gilbert also reported other bed features—'anti-dunes'—at still higher stresses. But these only occurred when the Froude number of the fluid flow exceeded unity. They probably result from a special type of instability due to an interaction between the local bed profile and local fluid current fluctuations arising from standing surface water waves. Gilbert's description on his p. 32 tallies.

Plot VIII, for which the bed was smooth and unrippled, has not been extended downwards as far as $\Theta = 0{\cdot}4$ because the bed drag became too small to measure precisely. But large ripples which formed when Θ was near the threshold—estimated at $\Theta_{(t)} = 0{\cdot}3$ at a mean flow velocity of 3·3 cm/s—also disappeared suddenly when the flow velocity was raised to 5·5 cm/s (Bagnold 1955). The disappearance should therefore correspond to $\Theta \sim 0{\cdot}3\sqrt{(5{\cdot}5/3{\cdot}3)} = 0{\cdot}39$.

The critical stress $\Theta = C_- \tan\alpha_-$ first emerged in § 3*a* (ii) in the derivation of the unit stress γ. It was assumed in § 3*c* that the critical stress $C_- \tan\alpha_-$ represents (i) the maximum possible yield strength of any exposed bed-surface grains by virtue of their own immersed weight and of their static friction coefficient $\tan\alpha_-$, and (ii) the coincident stress at which the fluid shear resistance τ_0' ceases to be significant on account of increasing grain concentration. Both these assumptions were hypothetical only.

But the above-mentioned disappearance of the bed ripples provides a third piece of confirmation of the real existence of this critical tangential stress; the other two being involved respectively in the satisfactory evaluation of the number G in § 5, and in the correct prediction in § 7*d* of the stress at which a suspended load should become fully developed. In all three cases experiment indicates a value close to 0·4.

(e) Consistency of the transport-rate results for wind-blown and water-driven grains

Two other basic assumptions attract comment. The apparent equality of the constant A in (42) for wind-blown and water-driven grains, as shown in figure 16, rests on the assumption in § 7 *a* that B_b as given by (41) should be halved for wind-blown grains, on the grounds that $\tan\alpha_0$ in (40) should in this special case be replaced by the static stress ratio $\tan\alpha_-$, which is about double the experimental dynamic ratio $\tan\alpha$. This assumption may be thought rather arbitrary. But by another basic assumption made in § 3 *a* (ii), following the argument in § 2 *a*, the bed load is supposed exerted via the normal grain stress P entirely on the bed grains and not at all on the static fluid between them. This may also be doubted in the case of liquids, whose density is comparable with that of the grains. Then let both assumptions be denied, the first for wind-blown grains and the second for water-driven grains. As an alternative to the second let it be assumed that the stress P in liquids is exerted at the boundary equally on the bed grains and on the static fluid between them, both in the present gravity context and also, necessarily, in the rotating-drum experiment in which $\tan\alpha = T/P$ was measured. Then this ratio, as measured in a liquid, must be doubled when applied to wind-blown grains, and B must therefore be halved. The two alternative assumptions seem therefore to be merely different ways of regarding the same effect.

10. Transport of heterogeneous grains

For simplicity let the grains vary in size only. If δq is the proportionate volume or weight of any constituent grade as isolated within a narrow diameter range δD the grading of the mixture can be defined by the curve of $p' = \delta q/\delta D$ against D, where $q = \int p'\,\mathrm{d}D = 1$ for a complete sample. Since the interval δD, e.g. between sieve sizes, is arbitrary and can be made constant from grade to grade, the ordinate p'_n for any grade D_n can be replaced by $p_n = p'\delta D_n = q_n$. It is convenient to plot $\log_{10} p$ against $\log_{10} D$. Otherwise the extreme grades cannot be adequately represented.

Let the grading of the mixture be represented by any arbitrary curve such as that of figure 18 *a* which has a single maximum.

Let the mixture have been previously deposited at random to form a stationary grain bed. It is supposed that the finer grains have to some small but appreciable extent rolled down by gravity into positions between the coarser ones, and to this extent are relatively less exposed to an applied fluid stress $\mathscr{T}_0$. The applied stress is therefore not uniformly distributed over the surface, the coarser grains receiving a disproportionately large share.

As $\mathscr{T}_0$ is gradually increased some grains of diameter D_n will begin to move. The initial threshold stress $\mathscr{T}_{tn}$ at which they do so may be expressed by

$$\mathscr{T}_{tn} = (\sigma-\rho)\, g\Theta'_{tn} D_n, \tag{54}$$

where Θ'_{tn} is some modified value of the experimentally ascertainable parameter Θ_t for the whole mixture.

Both Θ_t and Θ'_{tn} depend on two factors: (i) the kind of force applied to the individual grains; if the force is a fluid force only, as at the initial threshold of movement, and if the fluid is either fully turbulent or fully laminar, this factor should remain constant from grade to grade; (ii) the geometrical relationship of the individual grain to its neighbours on the bed surface; this factor will include any mal-distribution of (i) as between grains of different sizes; but since D is already cared for in (54), only the relative frequency p_n should be involved.

For a uniform bed in which D_n is constant and $p_n = 1$, $\Theta'_{tn} = \Theta_{tn}$. The effect of adding a small proportion p_{n+} of bigger grains D_{n+} scattered over the surface would effect a slight increase in Θ_{tn} to Θ'_{tn} for the original uniform grains because they are now slightly sheltered. But Θ_{n+} for the few coarse grains would be considerably reduced to Θ'_{n+}, since these grains take considerably more than their share of the applied stress. As the proportion p_{n+} of the coarse grains is increased however the relative reduction in Θ_{tn+} would become progressively less, till when $p_{n+} = 1$ and $p_n = 0$, Θ'_{tn+} coincides with Θ_{tn+} and Θ'_{tn} becomes infinite.

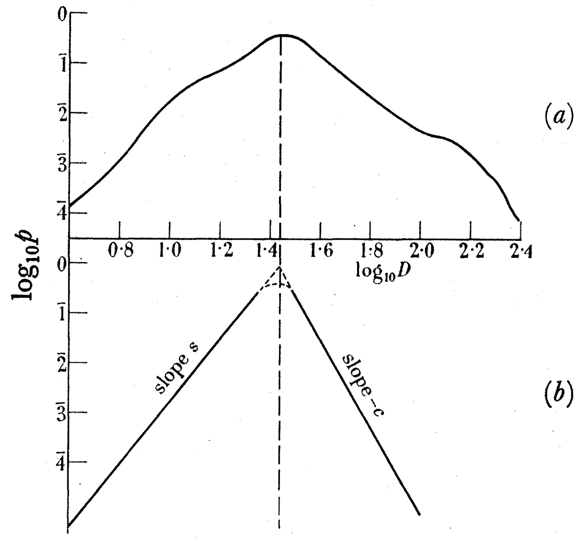

FIGURE 18. Plot of $\log_{10} p$ against $\log_{10}$ grain diameter. The dashed vertical line indicates the dominant diameter D.

Exchanging n and n_+ for the similar addition of an increasing proportion of smaller grains to a uniform surface, the effects will be inverse.

Introducing some dominant diameter $D_@$, the two effects can be represented by

$$\left.\begin{aligned} D_n > D_@: \quad \mathscr{T}_{tn} &= (\sigma-\rho)\, g\Theta_t D_{n+} f(p_n), \\ D_n < D_@: \quad \mathscr{T}_{tn} &= (\sigma-\rho)\, g\Theta_t D_{n-}/f'(p_n), \end{aligned}\right\} \tag{55}$$

where f and f' are both direct functions of p_n and should be independent of D.

Now over any reasonably large bed area $D_{n+}f(p_n)$ and $D_{n-}/f'(p_n)$ will have constant common values at a number of local spots on the surface. So erosion will occur simultaneously at all such spots. Moreover, if the eroded grains are collected somewhere downstream their grading should be definable by

$$D_{n+}f(p_n) = D_{n-}f(p_n) = \text{constant}, \tag{56}$$

and if f and f' are independent of D the grading relative to the dominant diameter $D_@$ should be unique and independent of that of the parent bed.

If (55) is to be generally applicable $f(p_n)$ and $f'(p_n)$ must be such that each coincides with p_n at zero and at unity, in order to include the ultimate case of constant D. Hence simple power functions are indicated, which may be expressed by

$$f(p_n) = p_n^{1/c} \quad \text{and} \quad f'(p_n) = p_n^{1/s},$$

where c and s, which might well be equal, should be attenuation constants depending only on the general conditions, e.g. turbulent or laminar fluid flow, the general value of G as determining the type of saltation, etc. and probably grain shape.

Accordingly the grading of the removed grains, and therefore of the load carried away downstream, should be given by

$$\log p/p_{@} = -c \log (D_{n+}/D_{@}) \qquad (57a)$$

$$= s \log (D_{n-}/D_{@}), \qquad (57b)$$

where $D_{@}$ should coincide with the dominant grain diameter of the bed surface as it may be constituted at the moment. In other words, when $\log p$ is plotted against $\log D$ (due regard having been paid in the analysis to the minute quantities of the extreme grades), the result should approximate to two straight lines intersecting at $D_{@}$ as in figure 18*b*; whereas if the diameter scatter were due merely to random error the result should approximate to a parabola.

The above appears to provide a physical explanation of the surprisingly consistent experimental results found for wind-blown sands (Bagnold 1941, chap. 10). The relative proportions p of coarse grains D_+ present in the sand removed from beds of known grading were found to attenuate strictly according to (57*a*). The coefficient c had a constant value of 9 irrespective of any excesses of coarse grades present in the parent bed. Any such excesses were left behind on the bed surface, which became progressively coarser. Continued erosion required a progressive increase in $\mathscr{T}_0$. $D_{@}$ increased progressively, but c remained unchanged.

A difficulty arises, however, in the case of the smaller grains. For these cannot be similarly left behind to accumulate unsheltered on the bed surface as $\mathscr{T}_0$ is increased. Experimentally the grading of the removed grains tended, on the side D_-, to conform to that of the bed. It is uncertain, however, whether or not if the moving load of eroded grains had been allowed to travel on downstream over an indefinite length of the same bed, the excess of small grains would have been progressively removed from the load by selective deposition. Samples of sand taken from the crests of dunes usually give $s \sim c \sim 9$.

No similar quantitative work is traceable for the more important case of water-driven grains, but the foregoing seems generally applicable.

The following inferences may be drawn tentatively:

(i) Provided the source of an experimental flow of grains is an erosion bed upstream, the flow should remain steady, and its grading constant with distance, under a constant applied stress, if the grading of the parent bed conforms to (57) with c and $s >$ some minimum value. And the results of the preceding sections should be applicable as for $D = D_{@}$.

(ii) Equation (57) applies to the grain load per unit bed area moving away from the erosion bed. If an attempt is made to imitate the conditions by introducing artificially a feed of grains having the same grading as the bed, the results may not be the same. For the feed has necessarily the nature of a transport rate. And the moving load will not have the same

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grading as the bed over which it is passing unless the constituent grades all have the same travel velocities $\overline{U}_n$. According to (44) $\overline{U}_n$ for the bed load at any rate should vary as B_n which is a direct function of D_n. It is possible therefore that a self-adjustment of the grading to that of the bed surface may take place along the flow, with consequent differential depositions and erosions. If so the texture of the bed surface may never remain constant along the flow, and steady conditions may never obtain.

(iii) A relation analogous to (57) is possible for the case of constant grain size and varying density if it can be assumed that gravity causes a tendency for relatively denser grains to oust the relatively less dense upwards from the mean bed surface, so that the latter receive more than their share of the applied stress. $(\sigma-\rho)_n$ could then be substituted for D_n in (57).

(iv) The results for uniform grains would be quite inapplicable in cases where the medial grades which should dominate are absent or are so deficient that the grading curve of the source grains has two widely separated maxima. For over a considerable range of Θ the larger grades being immovable would constitute a partially fixed bed. This is beyond the bounds of the stream case. $\mathcal{T}_t$ and A in (42), and the nature of the turbulence as it affects grain suspension, are all indeterminate (see §12*a*).

PART III. CONDITIONS BORDERING THE STREAM CASE

The relations found in part II can be expected to hold only if the conditions conform to the following requirements:

(i) The fluid flow is turbulent.

(ii) The grain concentration attenuates to zero within the fluid flow depth.

(iii) The bed is everywhere composed of stationary but erodible and dispersible (cohesionless) grains.

(iv) Conditions resulting from the application of constant tractive forces must themselves remain constant with time. That is, having attained maturity the bed surface suffers no further change.

It will be useful to examine briefly the effects of non-fulfilment of each of these bordering conditions.

11. Laminar fluid flow

(*a*) *Infinite flow depth*

No experimental work appears to have been published on the movement of grains in non-inertial (laminar) fluids. But according to the evidence outlined in §1*b* of the rotating-drum experiment both the grain and the fluid elements in the shear resistance should be proportional to the rate of shear in any system for which the number $G^2<100$, i.e. when the grain motion is also non-inertial.

Assuming the approximate validity of extrapolating Einstein's relation $\eta'/\eta = 1+\frac{5}{2}C$ to high concentrations as regards the shearing of the intergranular fluid alone (§1*b*)

$$\tau' = (1+\tfrac{5}{2}C)\,\eta\frac{\mathrm{d}u}{\mathrm{d}y} = f(C)\,\eta\frac{\mathrm{d}u}{\mathrm{d}y}. \tag{58}$$

And from (5*b*) we have for the shearing of the grain dispersion

$$T = 2{\cdot}2\lambda^{\frac{3}{2}}\eta\frac{\mathrm{d}U}{\mathrm{d}y} = F(C)\,\eta\frac{\mathrm{d}U}{\mathrm{d}y}. \tag{59}$$

Writing $T = P\tan\alpha = \gamma\tan\alpha\int_h C\,\mathrm{d}h$, and $\mathrm{d}u/\mathrm{d}y = \mathrm{d}U/\mathrm{d}y + \mathrm{d}\mathscr{U}/\mathrm{d}y$, where $\mathscr{U}$ is the relative velocity in the flow direction, both viscosity and rate of shear can be eliminated, so that

$$\frac{\mathscr{T}}{T} = \frac{T+\tau'}{T} = \frac{\Theta}{\tan\alpha\int_h C\,\mathrm{d}h} = \frac{F(C)+(1+\mathrm{d}\mathscr{U}/\mathrm{d}U)f(C)}{F(C)}, \tag{60}$$

where Θ is the total applied tangential stress at any plane h as given by (10).

$\mathrm{d}\mathscr{U}/\mathrm{d}U$ arises from the variation of the drag coefficient ψ' with concentration (§ 6 *b*). And by the approximation $\psi' = \psi(1-C)^{-3}$, $\mathscr{U}$ can be written

$$\mathscr{U} = \sqrt{\left(\frac{4\gamma\tan\alpha}{3\rho\psi}\right)}\,(1-C)^{\frac{3}{2}}, \tag{61}$$

whence
$$\mathrm{d}\mathscr{U}/\mathrm{d}U = \frac{\mathrm{d}\mathscr{U}/\mathrm{d}y}{\mathrm{d}U/\mathrm{d}y} = -\frac{F(C)\sqrt{\left(\dfrac{3}{\rho\gamma\tan\alpha}\right)}\,(1-C)^{\frac{1}{2}}\,\mathrm{d}C/\mathrm{d}h}{\int C\,\mathrm{d}h}.$$

From a few numerical evaluations by the method outlined below it seems that $\mathrm{d}\mathscr{U}/\mathrm{d}U$ in (60) can be ignored as being small, cf. unity.

Putting
$$\zeta = \frac{f(C)}{F(C)+f(C)} = \frac{\text{fluid shear stress}}{\text{whole shear stress}},$$

(60) gives
$$\int C\,\mathrm{d}h = \frac{\Theta(1-\zeta)}{\tan\alpha}, \tag{62}$$

and
$$\frac{\tau'}{\gamma} = \Theta\zeta. \tag{63}$$

Values of ζ in terms of C are shown in figure 19.

For the boundary conditions τ_0'/γ becomes the experimental threshold stress $\Theta_{(t)}$, which may be given White's (1940) value 0·2, whence ζ_0 is obtained from (63) for any given stress Θ. The whole load on the bed surface $\int_0 C\,\mathrm{d}h = \chi_0 = \dfrac{\Theta-\Theta_{(t)}}{\tan\alpha}$ by (20).

If the fluid flow depth is sufficient for Θ to be taken as approximately constant within the grain-occupied zone, (62) can be solved by a simple graphical method, giving C a range of arbitrarily decreasing values from C_0 and taking the corresponding values of ζ from figure 19.

For example, putting $\Theta = 2$, $\Theta_{(t)} = 0{\cdot}2$, $\tan\alpha = 0{\cdot}75$, the function ζ_0 at the bed surface is found to be 0·1 and the whole load χ_0 is 2·4. For these values the profile of C against the distance h is that shown in figure 20. The height of the centre of gravity of the dispersion is given by

$$1-\zeta_{\text{c.g.}} = \tfrac{1}{2}(1-\zeta_0),$$

whence $\zeta_{\text{c.g.}} = 0{\cdot}55$, and from figures 19 and 20 the concentration $C_{\text{c.g.}} = 0{\cdot}025$ and occurs at a height of 16 diameters. The concentration profile is independent of the viscosity. Compared to the movement of the bed-load grains under turbulent fluid flow, that under laminar flow appears to be characterized by a very high degree of dispersion.

From (58) and (63) the velocity at any distance h from the boundary becomes

$$u = \frac{\Theta\gamma D}{\eta}\int_0^h \frac{\zeta}{f(C)}\,\mathrm{d}h = \frac{\Theta\gamma D}{\eta}\int_0^h \frac{(1-\zeta)}{F(C)}\,\mathrm{d}h, \tag{64}$$

and the velocity profile for $\Theta = 2$ is as shown in figure 20. The relative velocity $\mathscr{U}$ enters as a constant (approx.) of integration.

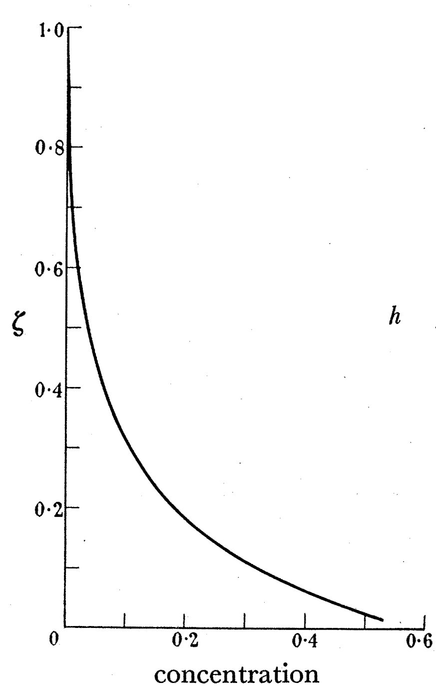

FIGURE 19

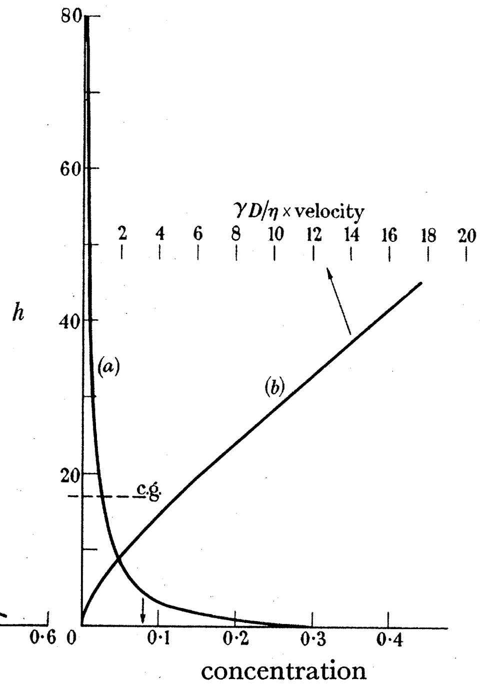

FIGURE 20

FIGURE 19. Curve of ζ against C, where

$$\zeta = \frac{\text{fluid shear stress}}{\text{whole shear stress}}, \quad \text{assumed} = \frac{1+\frac{5}{2}C}{2{\cdot}2\lambda^{\frac{3}{2}} + (1+\frac{5}{2}C)},$$

when both fluid and grain flows are non-inertial.

FIGURE 20. (a) Calculated concentration, and (b) velocity, in laminar fluid flow for a constant applied stress $\Theta = 2$.

(b) *Limited flow depth; 'slurries'*

When the gradient $\mathrm{d}\Theta/\mathrm{d}h$ of the applied tangential stress is constant and finite, (62) is satisfied when C and therefore ζ are constant from top to bottom. Differentiating,

$$\begin{aligned}\frac{C\tan\alpha}{1-\zeta} = -\frac{\mathrm{d}\Theta}{\mathrm{d}h} &= -\frac{\mathrm{d}\Theta_F}{\mathrm{d}h} + C\tan\beta \text{ (by (10))}\\ &= \tan\beta\left(\frac{\rho}{\sigma-\rho}+C\right) \text{ for parallel flow in open channels}\\ &= -\frac{1}{(\sigma-\rho)\,g}\,\frac{\mathrm{d}p}{\mathrm{d}x} \text{ for horizontal two-dimensional ducts.}\end{aligned} \tag{65}$$

This kind of grain flow at uniform concentration in the absence of fluid turbulence has several points of interest. It is presumably identifiable with the flow of a 'slurry', in which the mode of maintenance of the grain dispersion has not previously been explained.

Such a flow should only be possible when the grain movement is non-inertial, i.e. when $G^2 < 100$. At any plane distance $h_a - h$ below the plane h_a of zero Θ, the grain stress T is given by $\gamma C(h_a - h)$, and $G^2 = \sigma D^2 T/\lambda\eta^2$ (§ 1 *b*) becomes

$$G^2 = \frac{\sigma(\sigma-\rho)\,D^3 g C(h_a - h)}{\lambda\eta^2}, \tag{66}$$

which is greatest at the depth h_0 of the bed surface. For grains of density 2·65 (quartz), in water, uniform concentration at, say, $C = 0{\cdot}4$ should become possible in a channel 20 cm deep when $D < 55\,\mu$.

It is, however, much more easily observable in detail when D is large and $(\sigma - \rho)$ is made very small. In a turbulent water flow 6 cm deep (Bagnold 1955) the moving load of dispersed grains 1·36 mm diameter, for which $(\sigma-\rho)$ was only 0·004, was progressively increased. At a certain stage the turbulence began to be suppressed, being damped out by the increasing overall shear resistance. And on a further increase in the grain population the turbulence vanished altogether. As the turbulence faded out the concentration close to the bed surface which had previously been at its maximum was seen to decrease progressively; and that at the top of the flow increased. When the turbulence finally ceased, C was apparently uniform from top to bottom, at about 0·3. Had the fluid been more viscous uniformity of concentration would presumably have occurred at a lower value. Having attained uniformity in the now laminar fluid flow, C could be increased nearly to the mobile limit of about 0·53. Ultimately the whole flow 'froze' simultaneously at all depths.

Figure 19 shows that as C approaches its mobile limit ζ becomes very small. It should be noted that if ζ is neglected (65) becomes identical in form with the limiting condition (18) at which a grain bed (also supposed of uniform concentration) ceases to be stationary. So the simultaneous 'freezing' of the above flow appears to be merely the reverse of the simultaneous shearing of a static bed.

It may also be noted that for any given system C is uniquely determined by the flow conditions according to (65); and in an open parallel channel flow at any given concentration is possible at one gravity slope only.

At constant concentration (64) becomes

$$u = \frac{\gamma D\zeta}{\eta f(C)}\int_0^h \Theta\,\mathrm{d}h \tag{67}$$

and $\Theta = \dfrac{h_a - h}{h_a}\Theta_0$, whence the whole discharge is given by

$$\int_0^{y_a} u\,\mathrm{d}y = D\int_0^{h_a} u\,\mathrm{d}h = \frac{1}{3}\frac{\gamma\zeta y_a^2}{\eta f(C)}, \tag{68}$$

and the grain transport rate Q is σC times this, where C and thence $f(C)$ and ζ are determinable by (65).

It should be realized, however, that the experimental evidence of (5 *b*) which defines $F(C)$ and ζ is limited to fairly high concentrations only. Extrapolation beyond say $\lambda = 1$ or $C = 0{\cdot}1$ may not be justified.

The foregoing kind of flow of mutually inactive grains in a Newtonian fluid appears to constitute the borderline of rheology. In the simplest instance of an isotropic attractive force between grains an initial shear rate is necessary to create a sufficient dispersive stress P to overcome it. And with increasing attractive force the general bedward effect of gravity becomes progressively less significant.

When with larger grains G^2 appreciably exceeds 100, the grain and fluid flow characteristics are no longer similar, and the foregoing treatment is inapplicable. Flow at uniform concentration is unlikely to occur.

(c) *Comparison with turbulent fluid flow*

An expression analogous to (60) can be written for the other case of similar flow characteristics, that in which inertia effects dominate in both phases ($G^2 > 1500$ in a fully turbulent fluid). The fluid resistance is now

$$\tau' = l'^2 (\mathrm{d}u/\mathrm{d}y)^2, \tag{69}$$

where l' is some abstract quantity having the dimension of a length, which becomes the conventional 'mixing length' where the grain concentration disappears. The grain resistance as given by (5*a*) is

$$T = a\sigma(\lambda D)^2 (\mathrm{d}U/\mathrm{d}y)^2,$$

whence

$$\left.\begin{aligned} \frac{\Theta}{\tan\alpha \int C\,\mathrm{d}h} &= \frac{a\sigma\lambda^2 + \epsilon^2\rho(l'/D)^2}{a\sigma\lambda^2} \\ \text{and} \qquad \zeta' &= \frac{\epsilon^2\rho(l'/D)^2}{a\sigma\lambda^2+\epsilon^2\rho(l'/D)^2}, \end{aligned}\right\} \tag{70}$$

where $\epsilon = (1 + \mathrm{d}\mathscr{U}/\mathrm{d}U)$, which may not in this case approximate to unity. Putting $l' = \kappa' y = \kappa' Dh$, there might, on dimensional grounds, be a presumption that κ' is a function of λ only.

There seems to be no positive evidence that the fluid in this case is ever capable of carrying grains in suspension, which may simplify matters. But any approach to finding a relation between C and h analogous to that in subsection *a* is closed at present by lack of knowledge of l'.

Since l' is a variable, it seems impossible that any solution of (70) would allow of grain flow at a uniform concentration.

Comparing the effects in laminar and turbulent fluid flow, the important difference is apparent that whereas in laminar flow the resistance coefficient $(1+5C/2)\,\eta$ has a positive gradient with regard to C, the corresponding gradient $\partial l'/\partial C$ in turbulent flow is evidently negative owing to the suppression of the turbulence by the increasing grain resistance. This rather suggests that $\mathrm{d}\tau'/\mathrm{d}h$, which near the bed surface is positive in both cases (h being measured upwards from the surface), will, for the same load and the same total applied stress, be much larger in the turbulent case. If a laminar fluid were to become turbulent, it looks as if the highly dispersed load would tend to collapse on to the bed surface. Considering the bed-load grains alone, since $\int C_b\,\mathrm{d}h = \mathscr{T} - \tau'$

$$C_b = -\frac{\mathrm{d}\mathscr{T}}{\mathrm{d}h} + \frac{\mathrm{d}\tau'}{\mathrm{d}h}, \tag{71}$$

when $\mathrm{d}\mathscr{T}/\mathrm{d}h$, which can never be positive, can be supposed constant very close to the bed surface. It would conform with experience to suppose that in a turbulent fluid the effect of

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the negative value of $\partial l'/\partial C$ is to make $d\tau'/dh$, and therefore C_0 at the bed surface, as large as possible compatible with the magnitude $\int C\,dh$ of the whole available load, i.e., to allow of the minimum degree of dispersion (in the absence of suspension). For instance, C_0 was found in part II to reach its maximum of about 0·53 in a turbulent liquid when Θ_0 is only about 0·4. The whole bed load χ_{b0} is then little more than will just cover the bed-surface grains. Whereas if the theory is sound C_0 (figure 20) in the laminar case has only reached 0·3 at a stress $\Theta_0 = 2$ or five times as great.

12. Absence of a stationary grain bed

(a) *Fixed beds*

All the relations so far obtained have been restricted to the essential condition that the boundary to the flow shall consist everywhere of a bed of stationary but potentially mobile grains. Local erosion thereby enables any increase in the applied tangential stress to be offset by the extra resistance of an increased bed load. Under this condition the fluid element τ_0'/γ in the resisting boundary stress can never rise above its threshold value. And by the preceding inferences in the case of a turbulent fluid the concentration C_0 at the boundary should be the maximum compatible with the magnitude of the load as set by (19). The load is the maximum possible, and the flow may be conceived as 'saturated'.

Suppose, however, that though (18) is satisfied, so that a stationary grain bed can exist, this bed is allowed to erode away till a patch of underlying fixed floor becomes exposed. Over this patch no bed grains are available to maintain the above equilibrium. So τ_0' may increase. Since the basic bed-load relation $T/\gamma = \chi_{b0}\tan\alpha = \Theta_0 - \tau_0'/\gamma$ of (19) must still hold, there must be a corresponding decrease in the bed load χ_{b0} and a decrease therefore in the concentration C_0 at the boundary. Once this has occurred, it seems likely in the case of a turbulent fluid to permit of a radical readjustment of the mutual distributions of concentration and rate of shear, with greatly increased flow velocities close to the boundary and a marked rise in the centre of gravity of the grain load. It is possible therefore that a small decrease in the load may be accompanied by such a large increase in the mean travel velocity of the grains that their transport rate is increased.

Such changes are confirmed by the experimental evidence. Recent velocity measurements (Bagnold 1955, p. 200) close to the boundary disclose a sudden marked increase of velocity as occurring concurrently with an exposure of the fixed floor. The transport rate of wind-blown grains has been found (Bagnold 1941, chap. 5) to be more than doubled. And Gilbert (1914) reported a similar effect on fine grains in water when their bed contained a proportion of much larger and less mobile grains.

In terms of expression (42) this increase in the transport rate is attributable partly to an increase in the constant A due to the higher centre of gravity of the grains, and partly to a decrease in the drag coefficient ψ' in the expression for B brought about by a general decrease in the concentration near the boundary.

A result of this increase in the transport rate should be that under a constant applied stress continued erosion takes place at the edges of an exposed patch of fixed floor, causing the patch to grow bigger. Consequent deposition on the grain bed downstream between such bare patches should tend to maintain it as a series of persisting mounds. This effect is very noticeable in laboratory flume experiments. A grain bed never erodes away evenly.

As long as the source of available grains is unlimited the flow is still 'saturated' though the transport rate may be larger. But if the source is limited, the flow may be made unsaturated to any arbitrary extent. If in a closed (circulatory) system the total quantity of grains present is conserved, the load χ remains constant however much Θ_0 is increased, and the fluid boundary resistance τ_0'/γ increases with Θ_0.

It should be noted particularly that unless a mobile grain bed exists everywhere the entire basis is invalidated of any inferences such as those drawn in § 7 *d* relating the suspended load to the bed-load transport rates. For when the boundary is partly fixed and partly moving both it and the nature of the turbulence over it become quite undefinable. In unsaturated grain flow it is clearly possible for a suspended load to be transported in the absence of any bed load at all, e.g. the atmospheric transport of a large load of suspended dust.

(*b*) *Flowing beds*

Consider the implications of (18) in § 3 *b*. When this criterion

$$C_- \tan\alpha_- > -\mathrm{d}\Theta/\mathrm{d}h = -\mathrm{d}\Theta_F/\mathrm{d}h + C_- \tan\beta$$

is satisfied a grain bed will remain stationary under any applied tangential stress however great, owing to the exertion of the bed load upon its surface. The whole concentration profile, including that of the grain bed, will be as sketched in figure 21 *a*, except in the special case of a slurry (§ 11 *b*), when figure 21 *b* should be representative.

But when the gradient $\mathrm{d}\Theta_L/\mathrm{d}h$ of the bed's ultimate yield stress coincides with the gradient $\mathrm{d}\Theta/\mathrm{d}h$ of the total applied tangential stress, a simultaneous shearing of the whole bed must take place to an indefinite depth. The bed will begin to flow.

In the special case of a slurry (very fine grains in laminar fluid flow) it appears from § 11 *b* that all the bed grains will then be dispersed upwards to fill the available space at a general uniform concentration. The bed surface at h_0 would coincide with the upper boundary to the flow and would lose its significance. The velocity profile, as given by (64 *a*), would be continuous, as sketched in figure 21 *d*.

But so long as the fluid remains turbulent there is no reason to suppose that the grains of the now moving bed would disperse to any greater extent than that needed to enable them to shear over one another. So the concentration within the bed should remain uniform, but at some very high value $C_@$ just lower than C_-. The bed surface would still exist at h_0 though it would now be in motion. It should continue to define the plane at which the concentration begins to attenuate, and just above which turbulence begins (if the Reynolds number of the flow above is adequate).

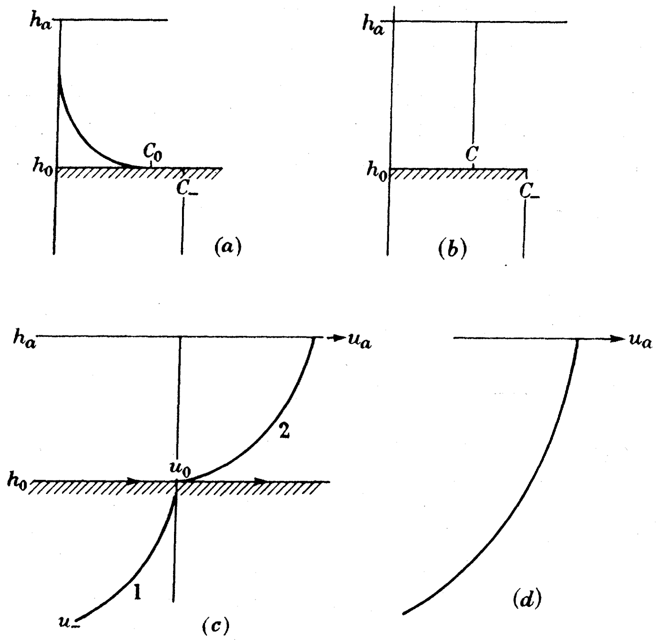
Within the flowing bed (18) has become the equation

$$C_@ \tan\alpha = -\frac{\mathrm{d}\Theta}{\mathrm{d}h}$$

$$= -\frac{\mathrm{d}\Theta_F}{\mathrm{d}h} + C_@ \tan\beta, \qquad (72)$$

where both $\mathrm{d}\Theta/\mathrm{d}h$ and $\mathrm{d}\Theta_F/\mathrm{d}h$ are constant within a homogeneous bed.

Under these conditions the relations found in part II for the stream case should still hold good above h_0 provided the grain dispersion above h_0 still attentuates to zero within the fluid. But all flow velocities will now be referable to the datum velocity of the bed surface. The transport rate of the grains above h_0 will thereby be increased; and to this rate must be added that of the flowing bed below. The velocity profile should show a kink at or just above h_0 indicative of the boundary to fluid turbulence and of the sudden fall in the concentration, as sketched in figure 21 *c*.



Concentration profiles; stationary beds; criterion (18) satisfied. (*a*) Turbulent flow (all cases); (*b*) laminar flow, $G^2 < 100$ (slurry case)

Velocity profiles; criterion (18) unsatisfied. (*c*) Moving bed; curve 1, constant C (max.); 2, attenuating C. (*d*) No bed; constant C (dispersed).

FIGURE 21

Ideally, no stationary grain bed could exist anywhere. But in reality it can. For since $C_@$ is constant, (72) constrains the bed flow to a definite gravity slope

$$\tan\beta = \tan\alpha + \frac{1}{C_@}\frac{d\Theta_F}{dh}, \tag{73}$$

depending only on $d\Theta_F/dh$ (which is essentially negative). So the bed flow can only extend downwards as far as the plane which, at this slope, is tangent to an immovable sill or an exposed bed floor somewhere downstream.

But if $\tan\beta$ is the general slope of a mountain torrent, a brief increase in $d\Theta_F/dh$ to an unusual peak value which just satisfies (73) will result in a sudden and perhaps catastrophic flow of the whole layer of previously accumulated debris, irrespective of its thickness, leaving behind only such debris as will fill any local pockets in the torrent floor.

A continuing bed flow is possible, e.g. when a flooded river converges into a narrow gorge. A flowing shingle bed having a sharply defined submerged surface has recently been detected by Mr T. F. H. Nevins in New Zealand (Bagnold 1955, correspondence on).

If a fixed floor lies at a known depth $-h_f$ below the surface h_0 of a flowing bed, the datum velocity of the surface can be obtained as follows. The total applied tangential stress at any plane $-h$ within the bed is represented by

$$\Theta_{-h} = \Theta_0 - h\,d\Theta/dh$$
$$= \Theta_0 - hC_@\tan\alpha \quad \text{(by (72))}.$$

And since the intergranular fluid resistance τ'/γ can be neglected at this high grain concentration, $\Theta_{-h} = T_{-h}/\gamma$. Hence according to the value of G^2 from (66) the empirical relations (5*a*) and (5*b*) yield

$$G^2 > 1500; \quad \sqrt{\left(\frac{\sigma}{\gamma}\right)}\frac{dU}{dh} = \frac{(a_i)^{-\frac{1}{2}}}{\lambda}(\theta_0 - C_@ \tan\alpha\, h)^{\frac{1}{2}}$$

and

$$U_0 = \tfrac{2}{3}(a_i C_@ \tan\alpha)^{-\frac{1}{2}}\{(\Theta_0 - C_@ \tan\alpha\, h_f)^{\frac{3}{2}} - \Theta_0^{\frac{3}{2}}\}, \tag{74a}$$

$$G^2\, 100; \quad \frac{\eta}{D\gamma}\frac{dU}{dh} = a_v^{-1}\lambda^{-\frac{3}{2}}(\Theta_0 - C_@ \tan\alpha\, h)$$

and

$$\frac{\eta}{D\gamma}U_0 = \frac{1}{2}\frac{a_v^{-1}\lambda^{-\frac{3}{2}}}{C_@ \tan\alpha}\{(\Theta_0 - C_@ \tan\alpha\, h_f) - \Theta_0^2\}, \tag{74b}$$

where a_i and a_v are the numerical constants of (5*a*) and (5*b*) respectively.

From the jerky and intermittent movement of the bed grains which is sometimes seen in the laboratory when the bed flows but slowly, it is probable that λ may then approximate more to the theoretical limit 22 than to the free flow limit 14 previously assumed. If so the coefficients may be rather larger (Bagnold 1954, p. 61) and $\tan\alpha$ may approximate to its static value. The resulting bed velocities would then be somewhat smaller.

Under parallel flow in an open water channel $-d\Theta_F/dh = \tan\beta \dfrac{\rho}{\sigma-\rho}$ (§ 3*a* (iii)). Whence by (73)

$$\tan\beta = \frac{C_@ \tan\alpha}{(\rho/\sigma-\rho)+C_@}. \tag{75}$$

If the bed remains in the above compact state $C_@$ may be taken as say 0·6 and $\tan\alpha$ as $\tan\alpha_- \sim 0{\cdot}63$. For grains of the density of quartz this gives a critical slope, in water, of 0·33 or $\beta = 18$ to 19°. But if at higher speeds the bed tends to disperse slightly to its 'fluid' limit $C \sim 0{\cdot}53$, $\tan\alpha$ for large inertial grains is likely to fall to 0·32, which would result in a critical gravity angle of only 8 to 9°. And if the intergranular water were heavily charged with silt or mud the effective value of ρ in (71) might rise from 1 to perhaps 1·8, with a further reduction of β to 3 or 4°.

When special light grain material is used in laboratory flume experiments the critical slope may become very small. For example, beds of polystyrene grains in water, $(\sigma-\rho) = 0{\cdot}06$, would flow at slopes between 0·01 and 0·02. The sometimes anomalous behaviour of such materials may thus be explained.

Relations (72), (73) and (75) are all independent of the depth h_a of flow above the bed surface. If h_a is zero, the bed flow becomes the creep of supersaturated soils.

When Θ_F is zero for a fluid at rest, or when $d\Theta_F/dh$ is very small, as under atmospheric flow of virtually infinite depth, (74) reduces to the condition for a simple avalanche, with $\tan\beta = \tan\alpha$.

In a horizontal closed duct $d\Theta_F/dh = \dfrac{dp/dx}{(\sigma-\rho)\,g}$. So by (18) no stationary grains could overlie the duct floor when

$$-dp/dx > (\sigma-\rho)\, g C_- \tan\alpha_-.$$

But if the friction coefficient between the bed grains and the floor were smaller than $\tan\alpha_-$ between the bed grains themselves the whole bed would slide bodily along the floor, as indeed it is sometimes found to do.

13. Effect of local variations in the gravity slope of a uniform stationary grain bed; dunes

Bed ripples of the kind dealt with in §4 should, according to the explanation given, be restricted to beds of low-inertia grains whose shearing is appreciably affected by fluid viscosity. And since the primary cause of such features concerns the shear strength of the individual bed-surface grains their linear scale is also likely to be restricted.

But in streams both of air and of water other, anomalous, features are found which are unrestricted either to small grains or to a small linear scale. The scale is indeed sometimes of an altogether different order of magnitude. These features, which for convenience may be distinguished as 'dunes', require some other and more general explanation.

Bed features of any kind essentially involve variations in the gravity slope $\tan\beta$ of the bed surface along the flow. The transport rate Q, whether by (42) or (46), is a function of $\tan\beta$, independently of whether the apparent applied stress $\mathscr{T}_F$ is also a function of $\tan\beta$. And any change δQ along the flow is in the two-dimensional case synonymous with an equal rate of deposition on to, or erosion from, the bed. If either persists in time it must cause a progressive change in the local bed slope. Thus interactions between $\tan\beta$ and Q along the flow may be such as to be mutually accentuating. In which case the flow will not remain steady, and a bed initially plane on a scale greater than that of ripples will not remain plane as time goes on.

The bed-load transport-rate expression (42*d*) can be written

$$Q_b = a\frac{\sec\beta}{\tan\alpha-\tan\beta}(\mathscr{T}_F-\mathscr{T}_t)\,\mathscr{T}_F^{\frac{1}{2}}, \tag{76}$$

where $a = A\dfrac{\sigma\tan\alpha\sqrt{(2\tan\alpha/3\psi)}}{\sqrt{(\rho)}\,(\sigma-\rho)\,g}$. Experimental evidence regarding the variation of the parameter $\mathscr{T}_t$ with the gravity slope is lacking, the slopes usually dealt with being very small. But since the inherent shear strength of the bed-surface grains is involved $\mathscr{T}_t$ seems likely to be a function of $\tan\beta$ of the form

$$\mathscr{T}_t = \gamma\Theta_t'(\tan\alpha_- - \tan\beta),$$

where Θ_t' should be a reasonably constant numerical coefficient.

Substituting this in (76), and ignoring any discrepancy between the static and dynamic values of $\tan\alpha$, the partial derivative $\partial Q/\partial\tan\beta$ is approximately

$$\frac{\partial Q}{\partial\tan\beta} = a\mathscr{T}_F^{\frac{1}{2}}\left\{\mathscr{T}_F\left(\frac{\sin\beta}{\tan\alpha-\tan\beta}+\frac{\sec\beta}{(\tan\beta-\tan\beta)^2}\right)-\gamma\Theta_t'\sin\beta\right\}. \tag{77}$$

Tan β may be assumed never to exceed $\tan\alpha$, both $\tan\beta$ and $\sin\beta$ become negative with β on local surfaces inclined upwards in the flow direction, but $\sec\beta$ remains positive. So $\partial Q/\partial\tan\beta$ by (77) is always positive and finite. For a given slope it is a minimum at the threshold value of $\mathscr{T}_F$. And for any given value of $\mathscr{T}_F$ it increases with increase in the downhill slope, so $\partial^2 Q/\partial\tan^2\beta$ is also always positive.

By the concept of Q in §7 as a measure of the rate of useful transporting work done it appears self-evident that under a constant impelling stress both Q_b and Q_s must vary in the same sense as the downward slope $+\tan\beta$. (Cf. the variation of the vehicular transport rate along a road of varying gradient.)

It follows that provided the grain flow is saturated immediately upstream of any point A on the bed, a decrease $-\delta\tan\beta$ in the slope of the bed beyond A will always tend to cause deposition on the decreased slope. And that the original change of slope will thereby become accentuated. Similarly, any local increase $+\delta\tan\beta$ in the bed slope will tend to cause erosion and a consequent further increase. Whether the deposition or erosion actually takes place will depend on how the local impelling stress $\mathscr{T}_F$ happens to be affected.

The apparent impelling stress $\mathscr{T}_F$ is a function of the whole fluid flow above the bed. It may, if the flow depth is large compared to the thickness of the bottom zone occupied by the dispersed grains, be taken as approximately independent of Q. And how far its local value in the immediate neighbourhood of the point A is affected by the local change of bed slope will depend on many other factors. These would include the whole form of the existing bed feature and its scale relative to the whole flow depth.

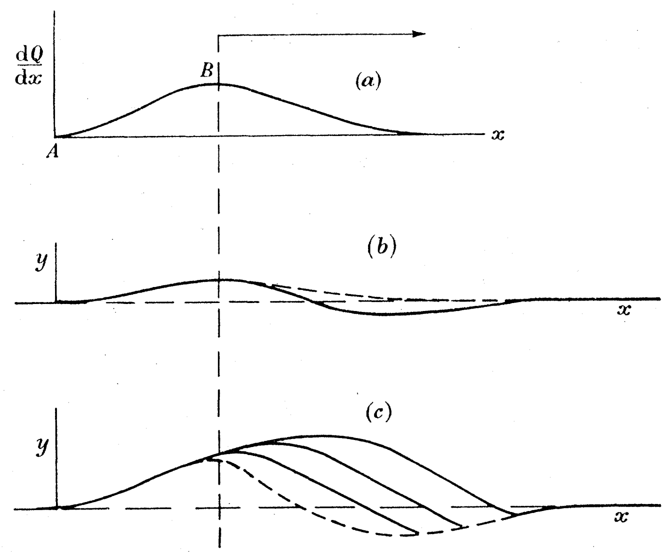


FIGURE 22

If the longitudinal profile of the bed surface is to remain constant, it would seem necessary in the two-dimensional case that the distribution of $\mathscr{T}_F$ over the surface should be such that everywhere

$$\frac{\mathrm{d}\mathscr{T}_F}{\mathrm{d}\tan\beta} = -\frac{\partial Q}{\partial\tan\beta}\bigg/\frac{\partial Q}{\partial\mathscr{T}_F},$$

and since $\partial Q/\partial\mathscr{T}_F$ is clearly always positive, $\mathrm{d}\mathscr{T}_F/\mathrm{d}\tan\beta$ must everywhere have an appropriate negative value.

The distribution of the applied fluid stress $\mathscr{T}_F$ over the longitudinal section of a bed feature of a given shape might be obtainable from the general dynamics of the fluid flow, or by measurement using a fixed model feature. But, except for a very large feature, serious complications are to be expected owing to an inevitable lag between a causatory change in $\tan\beta$ and a resulting change in Q. Depending on the flow velocity, the degree of dispersion of the grains and their relative inertia the deposition or erosion rate $\mathrm{d}Q/\mathrm{d}x$ per unit bed area will be distributed in an unknown way with regard to the distance x downstream. The distribution might be expected to rise to a maximum at some point B, as sketched in figure 22a.

The lag distance AB is likely to be a measure of the minimum possible ground scale of a dune. For wind-blown sand grains it seems to be of the order of several metres (Bagnold 1941, chap. 12). It may be still longer for fine water-suspended grains. But for water-driven grains unsuspended and moving very close to the bed surface the lag distance may well be so small as to overlap with the scale of primary ripples.

Between A and B the upward slope of a mound (figure 22*b*), resulting from continued deposition, should go on steepening until the deposition is arrested by a local increase in $\mathscr{T}_F$, the limiting upstream slope depending on the whole fluid flow conditions.

Beyond B, however, the effect of the influence of the mound on the fluid flow is likely to be more local. The downward lee face should steepen rapidly owing to the combination of deposition at B and erosion beyond due to the change of slope at B, which by (77) should exceed the deposition, $\tan\beta$ being larger. A stage may be reached, depending probably on the actual shape of the distribution curve (figure 22*a*), at which the fluid flow begins to break away. The local boundary stress may then be reduced so much that erosion on the lee face gives place to deposition (in effect $d\mathscr{T}_F/d\tan\beta$ becomes excessively negative at the summit B).

If and when this critical stage is reached the dune should be capable of extension downstream and of further increase in height (figure 22*c*) till the fluid flow conditions set some ultimate limit to the general scale. It seems noteworthy that under atmospheric flow of virtually unlimited depth the dunes formed are of apparently unlimited scale.

The idea that the grain flow *per se* tends inherently to form dunes on its bed, on an initial scale dependent on a certain 'lag distance', seems of some importance. For the ultimate balance set up between this tendency and its increasing inhibition by the fluid flow as the dune grows is likely to be very different in the three-dimensional case of a flow of unrestricted width.

In the two-dimensional case, supposed approximated to in a laboratory channel which is narrow relatively to the scale of the dune, the latter, if it is allowed by the fluid flow to grow at all, must do so as a transverse bar over which the whole fluid flow must pass. But experience shows that under a unidirectional flow of unrestricted width, alike of wind or of water, dunes tend to develop as isolated features disposed relatively to the flow in a diagonal pattern so that the fluid can divert and rejoin round each. The inhibitory increase in the fluid boundary drag on the rising upstream dune face must thereby be reduced, and the dune would thus be enabled to grow appreciably steeper and higher. The roughly diagonal pattern follows from the partial diversion of the grain flow below and its funnelling into channels between existing dunes; deposition would take place immediately downstream of such a funnel. The repeated diversion and rejoining of the fluid currents over the bed would be expected to introduce a new and special pattern to the internal flow of the whole fluid.

14. Conclusion

An attempt has been made in the foregoing paper to correlate for the first time a number of widely scattered phenomena connected with grain flow. The results appear to be mutually consistent and to afford simple explanations of much that has hitherto been somewhat mysterious. Quantitative deductions based ultimately on stresses measured between coaxial drums seem in reasonable agreement with such experimental data as can be found.

But a serious dearth of experimental data is very evident. The reasons for this have undoubtedly been that experiment in this field is laborious and often costly; precise measurement is peculiarly difficult; and in the absence of any adequate guide to the proper definition and relative significance of the many possibly measurable quantities involved, the choice of what to measure and of what details to record has been uncertain.

In such a case some over-stepping of the usual bounds of deductive reasoning may be justifiable if a coherent theory results which, however incomplete it may be, suggests new and more promising approaches.

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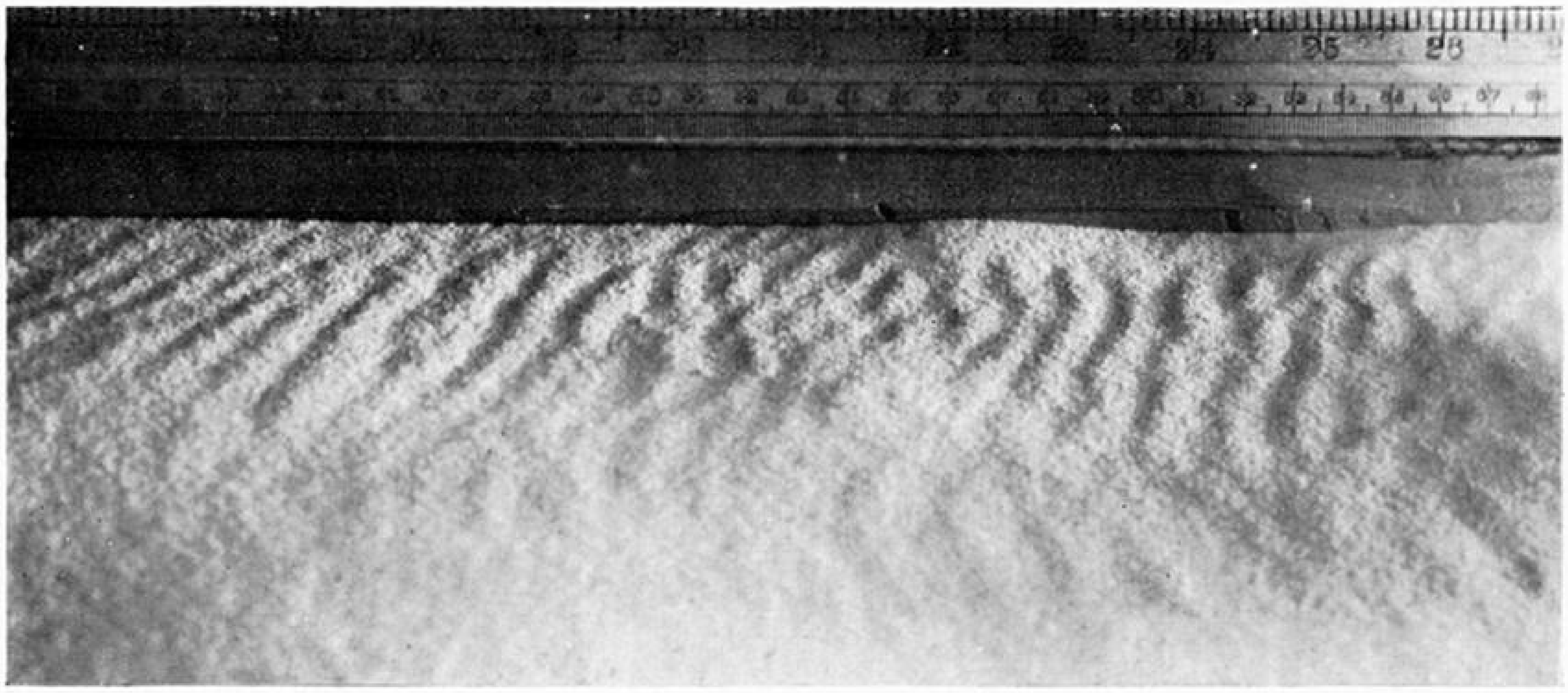
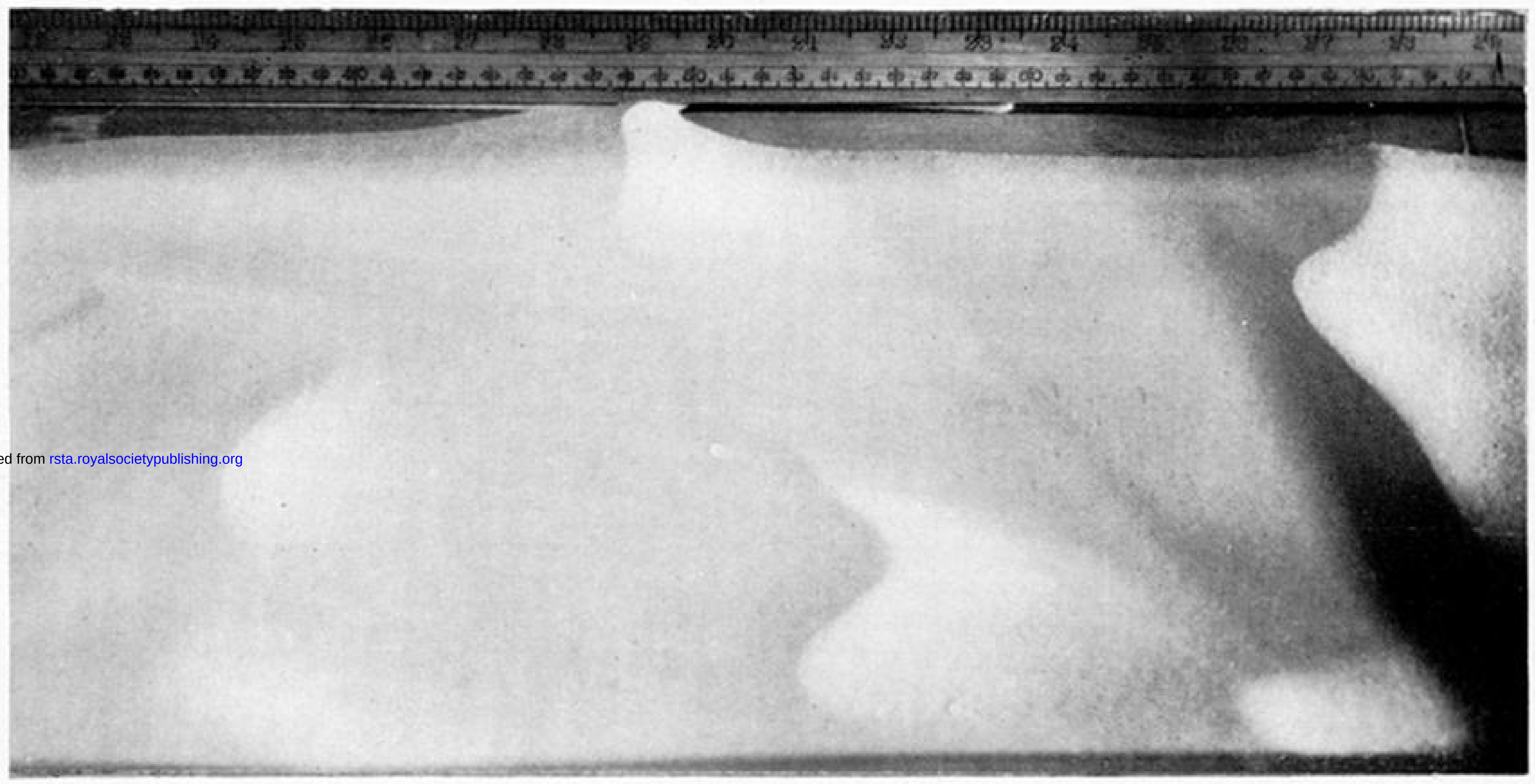
← wind direction

a



b



Effect of air viscosity on behaviour of very find wind-blown grains (diameter 80μ, quartz). Abrupt change in the bed features at the predicted critical surface stress. (*a*) Usual 'ballistic' ripples at low stresses. (*b*) Large (assumed secondary) features resembling those in water. (Reproduced from plate 10 of Bagnold 1941.)